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Drift-Independent Volatility Estimation Based on High, Low, Open, and Close Prices*

We present a new volatility estimator based on multiple periods of high, low, open, and close prices in a historical time series. The new estimator has the following nice properties: it is (*a*) unbiased in the continuous limit, (*b*) independent of the drift, (*c*) consistent in dealing with opening price jumps. Furthermore, it has the smallest variance among all estimators with similar properties. The improvement of accuracy over the classical close-to-close estimator is dramatic for real-life time series.

I. Introduction

Estimation of the volatility of a security is an important and practical issue in pricing options and measuring portfolio risks (Merton 1990). The classical estimator is based on the close-to-close prices only. More sophisticated estimators in literature use additional information such as high, low, and open prices to achieve better accuracy. However, in constructing these estimators, some assumed that the security price has no ‘‘drift’’ motion (such estimators tend to overestimate the volatility), while others assumed no opening price jumps (i.e., the opening price is the same as the previous closing price; such estimators tend to underestimate the volatility). In this article, we present a new unbiased estimator based on multiple periods of open, close, high, and low prices. Our estimator is independent of both the drift motion and opening jumps. The variance of our estimator is smallest among all estimators with similar properties.

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There is a large amount of literature on the modeling issue of the price movements of a security. We will not address such an issue in this article; rather, we simply assume that price movements can be modeled as a geometric Brownian motion, which means that the logarithm of the security price is a Brownian motion with two undetermined parameters, volatility σ and drift μ. The objective is to estimate the volatility based on the available price information (open, close, high, and low prices). It turns out that it is convenient to estimate the variance, which is defined to be volatility squared. We will deal with variance estimators in the rest of the article.

Following the work of Garman and Klass (1980), the price in each period of length T starts at the closing price of the previous period. Furthermore, each period is divided into two intervals with fractions f and $1 - f$. The trading is closed during the first interval of length fT; thus the price movement in this interval (before opening) is unobservable. The high and low prices in a data set are those observed from the second interval of length $(1 - f)T$ (trading interval). We mention that fT here is not the physical time interval during which markets are closed; rather, it is an effective time period that models the opening jump as an unobservable continuous price movement. The fraction f measures the relative size of the opening jump (compared with the price range of the continuous trading interval). The case $f = 0$ means that there is no opening jump, and the case $f \to 1$ implies that the price movement in the period is dominated by the opening jump. It should be pointed out that in general the value of f depends on the period length T. For example, the f that corresponds to the daily data will be greater than the one of the weekly data. We comment that the current model by Garman and Klass (1980) can be viewed as a special case of the jump diffusion model proposed in chapter 9 of Merton (1990). Here the drift and variance parameters corresponding to the continuous diffusion part are $\mu T(1 - f)$ and $\sigma^2 T(1 - f)$, respectively, the frequency parameter of the Poisson-driven jumping process is one, and the jump is assumed to be a Gaussian random variable with mean μTf and variance $\sigma^2 Tf$.

The notation adopted in the current article is similar to that used by Garman and Klass (1980):

- T = time interval of each period, which is set to one without the loss of generality;
- f = fraction of the period (between [0, 1]) that trading is closed;
- V = unknown variance, which is the unknown volatility squared (σ^2);
- C_0 = closing price of the previous period (at time 0);
- O_1 = opening price of the current period (at time f);

H_1 = the current period's high during the trading interval (between $[f, 1]$);
L_1 = the current period's low during the trading interval (between $[f, 1]$);
C_1 = closing price of the current period (at time 1);
o = $\ln O_1 - \ln C_0$, the normalized open;
u = $\ln H_1 - \ln O_1$, the normalized high;
d = $\ln L_1 - \ln O_1$, the normalized low;
c = $\ln C_1 - \ln O_1$, the normalized close.

Using the aforementioned notation, the classical variance estimator based on the close-to-close prices of an n-period historical data set can be written as

$$V_{CC} = \frac{1}{n-1} \sum_{i=1}^{n} [(o_i + c_i) - \overline{(o + c)}]^2, \tag{1}$$

where the subscript i denotes the quantity of the ith period, and $\overline{(o + c)} = (1/n)\sum_{i=1}^{n} (o_i + c_i)$. This estimator is independent of the drift μ and the opening jump f and is unbiased, which means $E[V_{CC}] = \sigma^2$, where $E[\ ]$ denotes taking the expectation. The classical variance estimator V_{CC} serves as a benchmark for all other variance estimators. We will later demonstrate that our new variance estimator preserves all the aforementioned properties of the classical estimator V_{CC} with the additional feature that the variance of our new estimator is much smaller.

The variance of an estimator measures the uncertainty of the estimation. The smaller the variance, the more accurate is the estimation. We say that an unbiased estimator A is more accurate than B if its variance is smaller than that of B, that is, $\mathrm{Var}(A) < \mathrm{Var}(B)$. Among a set of unbiased estimators, the estimator with the smallest variance will minimize the uncertainty of the estimation. Therefore, from the points of view of both theoretical consideration and practical application, it is desirable to find the minimum-variance unbiased estimator. In the seminal paper by Garman and Klass (1980), they found the minimum-variance unbiased quadratic variance estimator for a Brownian motion with zero drift. In this article we present a new minimum-variance unbiased quadratic variance estimator for Brownian motions with nonzero drifts.

One can reduce the variance of the classical close-to-close variance estimator by increasing the number of periods n. However, this is not a very viable option in practice, since a time series is rarely stationary over a long period of time; that is, its volatility could change slowly with respect to time. Thus, old information is of little relevance to the current situation. The alternative to improve accuracy other than increasing n is to use the other available information such as high, low, and open prices. Several works have been done along this line that we now review.

Parkinson (1980) found a variance estimator using the high and low prices only. It is

$$V_P = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{4\ln 2}(u_i - d_i)^2. \tag{2}$$

This estimator is only valid when there are no opening jumps ($f = 0$) and there is no drift ($\mu = 0$). Some empirical studies of V_{CC} and V_P on actual market data were given by Beckers (1983). A better variance estimator using the high, low, and close prices was found by Rogers and Satchell (1991) and Rogers, Satchell, and Yoon (1994). It is

$$V_{RS} = \frac{1}{n}\sum_{i=1}^{n}[u_i(u_i - c_i) + d_i(d_i - c_i)]. \tag{3}$$

V_{RS} is better than V_P in two different ways. First, the variance of V_{RS} is smaller than that of V_P—that is, $\mathrm{Var}(V_{RS}) < \mathrm{Var}(V_P)$; second, unlike V_P, V_{RS} is independent of the drift. Another important feature of V_{RS} is that V_{RS} equals zero when the security price makes a one-direction move, either $u = c$ and $d = 0$ for a straight-up move or $d = c$ and $u = 0$ for a straight-down move. This is because the price movements in such situations can be explained by the drift term alone (zero variance). However, V_{RS} still assumes no opening jumps ($f = 0$). Kunitomo (1992) also considered the case of nonzero drift and derived a variance estimator, but his formula is based on the extremes of a constructed Brownian bridge motion. Since the extremes of a Brownian bridge motion are unknown unless one has tick-by-tick trading data, Kunitomo's formula is of little use in practice. However, having tick-by-tick trading data is equivalent to having the full Brownian path (a discrete one); there is no need to use variance estimators based on extreme values in such a situation. Under the assumption of no drift ($\mu = 0$), Ball and Torous (1984) developed a numerical maximum likelihood variance estimator. Under the same restriction ($\mu = 0$), Garman and Klass (1980) derived analytically that the minimum-variance unbiased variance estimator is of the following combination:

$$V_{GK} = V'_O - 0.383V'_C + 1.364V_P + 0.019V_{RS}, \tag{4}$$

where the definitions for V'_O and V'_C are

$$V'_O = \frac{1}{n}\sum_{i=1}^{n} o_i^2, \tag{5}$$

$$V'_C = \frac{1}{n}\sum_{i=1}^{n} c_i^2. \tag{6}$$

We comment that the expression for V_{GK} shown in equation (4) was not explicitly given in Garman and Klass (1980). The closest formula in Garman and Klass to equation (4) is their formula 20, which depends explicitly on f. Although f may be inferred from the historical data set, it is not a direct market observable. Therefore, it is much preferable to have an estimator that is independent of f. A closer examination shows that the f dependency of formula 20 in Garman and Klass is spurious. Setting the weight coefficient a in their formula 20 to f leads to the expression given by equation (4), which is f independent. Notice that all three estimators V_P, V_{RS}, and V_{GK} are only valid under various assumptions (either no drift or no opening jumps). We now examine the biases caused by the no drift and no opening jumps assumptions.

The no drift assumption is a good approximation when the dimensionless parameter $\mu\sqrt{T}/\sigma$ is small. Time series for daily data usually satisfy this condition. However, it happens quite often in practice that the price of a security goes through a ''trendy'' phase, in which the drift could be large compared with the volatility. The trendy nature of a strong bull market and price movements of certain high-technology stocks in recent years are good examples of large drifts (at least during certain periods). Therefore, estimators V_P and V_{GK} will overestimate volatility during these periods. It is also clear that the no drift assumption may not be valid if the time period T is not small—for example, weekly or monthly time series. We now focus on the no opening jumps ($f = 0$) assumption. It is clear that the volatility caused by opening jumps is not reflected in the estimators V_P and V_{RS}, whereas it is included in the close-to-close variance estimator V_{CC}. Therefore, ignoring opening jumps will underestimate the volatility. We comment that one could remedy V_P and V_{RS} with regard to opening jumps by adding V_O' given by equation (5) to V_P and adding a similar term (V_O given by eq. [8] in the next section) to V_{RS}. Notice that the improved V_{RS} is still independent of the drift. However, the improved estimators do not have the property of minimum variance in their respective situations, namely, the improved V_P has a larger variance than that of V_{GK} given by equation (4) under the zero drift assumption, and the improved V_{RS} has a larger variance than that of the new estimator derived in the next section (V given by eq. [7]).

Drifts and opening jumps do occur in reality, and furthermore, since neither f nor μ is a market observable, it is desirable to construct an unbiased variance estimator that is independent of both f and μ. Notice that the estimators V_P, V_{RS}, and V_{GK} are all arithmetic averages of their corresponding single-period ($n = 1$) estimators. Therefore, they are single-period-based estimators, whereas V_{CC} is a multiperiod-based estimator. We now show that it is impossible to have a single-period-based variance estimator that is independent of both the drift μ and the opening jump f. We prove this result by contradiction. If such an estimator were to exist, we examine its behavior in the limit of $f \to 1$ (bearing in mind that the estimator is f independent). Under such a limit, we only have one nontrivial number o, namely the opening jump, whereas $u = d = c = 0$, since there is almost no time for the continuous random walk to take place. Clearly, the variance estimator based on one number o cannot have its expectation independent of the drift μ. Therefore, such a single-period-based estimator does not exist. An unbiased variance estimator independent of both the drift and the opening jump must be multiperiod based. We construct an estimator that possesses such properties and has minimum variance in the next section.

II. A Minimum-Variance Unbiased Variance Estimator Based on Multiple-Period Data

We now present our new variance estimator based on data of multiple periods ($n > 1$). We prove in appendix A that the minimum-variance unbiased variance estimator that is independent of the drift μ and the opening jump f must have the following form:

$$V = V_O + kV_C + (1 - k)V_{RS}, \tag{7}$$

where V_{RS} is given by equation (3) and V_O and V_C are defined as follows:

$$V_O = \frac{1}{n - 1} \sum_{i=1}^{n} (o_i - \bar{o})^2, \tag{8}$$

$$V_C = \frac{1}{n - 1} \sum_{i=1}^{n} (c_i - \bar{c})^2, \tag{9}$$

with $\bar{o} = (1/n)\sum_{i=1}^{n} o_i$, and $\bar{c} = (1/n)\sum_{i=1}^{n} c_i$. The constant k will be chosen to minimize the variance of the estimator V. Before determining k, we verify that V is indeed unbiased and independent of both the drift μ and the opening jump f. For each component, we have $E[V_O] = \sigma^2 f$ and $E[V_C] = E[V_{RS}] = \sigma^2(1 - f)$. Therefore, $E[V]$ equals σ^2, which means that the estimator V is unbiased. Since all three components V_O, V_C, and V_{RS} are independent of the drift μ, so is V. The fact that f does not appear in equation (7) shows that V is f independent. The f independency means that the new variance estimator is also valid when the quantity f is an independent random number (between zero and one) instead of a constant. Notice that all the aforementioned properties are satisfied for any given k.

We now determine the constant k to minimize the variance of V (which is equivalent to minimize $E[V^2]$). It is easy to find the solution for this quadratic minimization problem. The result is $k = k_0 \equiv E[(V_O + V_{RS})(V_{RS} - V_C)]/E[(V_{RS} - V_C)^2]$. An explicit expression for k_0 can be obtained by using the following results: (a) V_O is independent of

V_C and V_{RS}, since quantities from different intervals of a random walk are independent; (b) V_C and V_{RS} are uncorrelated, that is, $E[V_C V_{RS}] = E[V_C]E[V_{RS}]$, the proof of which is given in appendix B; (c) $E\,[V_C^2] = [(n+1)/(n-1)]\sigma^4(1-f)^2$, which is a result of the classical statistics; (d) $E[V_{RS}^2] = [(\alpha + n - 1)/n]\sigma^4(1-f)^2$, where $\alpha \equiv E[(u(u-c) + d(d-c))^2]/\sigma^4(1-f)^2$. We comment that since $E[(u(u-c) + d(d-c))^2]$ is proportional to $(1-f)^2$, α is independent of f; also α is always greater than one; otherwise the variance of the random variable $u(u-c) + d(d-c)$ would have been negative. The final expression for k_0 is

$$k_0 = \frac{\alpha - 1}{\alpha + \dfrac{n+1}{n-1}}. \tag{10}$$

For this value $k = k_0$, the variance of the estimator given by equation (7) reaches the minimum.

Although the expectation of $u(u-c) + d(d-c)$ is independent of the drift, the expectation of its square (α) does depend on the drift. Therefore, strictly speaking, the variance of the estimator given by equation (7) can only be minimized at a given drift, although V itself is independent of the drift. However, the effect of a nonzero drift on α is minor, as we now demonstrate. Rogers and Satchell (1991) showed that $\alpha \leq 2$ by using the triangle inequality. Our numerical calculation suggests that $\alpha < 1.5$ for all drifts. The quantity α reaches the minimum when the drift is zero. Using the formulae of moments provided by Garman and Klass, the value of α when the drift is zero is calculated to be 1.331. Thus, the maximum of α is only about 15% larger than its minimum, which means that the effects of the drift on α is minor. Since usually the drift is small for practical historical daily data, the estimator given by equation (7) should be optimized under the small drift situation. We suggest that α be set to the value 1.34 in practice.

Notice that k_0 is independent of f, but it does depend on α and the number of periods n. It reaches the minimum $(\alpha - 1)/(\alpha + 3)$ when $n = 2$, with the minimum numerical value being 0.076 (achieved with the minimum value of α 1.33), and it increases monotonically to the maximum $(\alpha - 1)/(\alpha + 1)$ when $n \to \infty$, with the maximum numerical value being 0.2 (achieved with the maximum value of α 1.5).

Assuming for the moment that there are no opening jumps ($V_O = 0$), then the estimators V_C and V_{CC} are equivalent. The two components V_{CC} and V_{RS} in equation (7) themselves are unbiased and drift independent variance estimators. The fact that k_0 can never reach zero or one shows that neither the classical close-to-close estimator V_{CC} nor the estimator V_{RS} alone has the property of minimum variance. The estimator with minimum variance is a linear combination of both V_{CC} and V_{RS}

with positive weights. Notice that the weight $1 - k_0$ on V_{RS} is always greater than that on V_{CC} (k_0), which reflects the fact that the variance of V_{RS} is smaller than that of V_{CC}.

Garman and Klass defined the efficiency of a variance estimator to be the ratio of the variance of the classical estimator V_{CC} to that of the current estimator. Thus, the higher the efficiency, the more accurate the estimator is for a given number of periods. By this definition, the efficiency of our new variance estimator given by equation (7) is

$$Eff = \frac{\mathrm{Var}(V_{CC})}{\mathrm{Var}(V)} = \frac{1}{f^2 + (1 - f)^2 k_0}, \tag{11}$$

where k_0 is given by equation (10). We see that the efficiency depends on the values of both f and k_0. The efficiency increases as the number of periods n decreases, while f remains fixed, since k_0 decreases as n decreases. Thus, the smaller the number of periods, the higher the efficiency. We examine the maximum and minimum efficiencies of our new estimator. At a critical value of f, that is,

$$f_c = \frac{k_0}{k_0 + 1}, \tag{12}$$

the efficiency reaches its maximum, which is (from eqq. [11] and [12])

$$Eff_c = 1 + \frac{1}{k_0}. \tag{13}$$

The highest efficiency is reached when $f = f_c$ and k_0 is at its minimum value 0.076 ($n = 2$, $\alpha = 1.331$). Under these conditions, the efficiency has the peak value of 14. This means that for these values of k_0 and f_c, the new estimator V using only 2-days' data (assuming that a period corresponds to a day) will have the same accuracy as that of the classical estimator V_{CC} using 3-week's data (only 5 trading days in a week). However, it is obvious that if the volatility is dominated by opening jumps ($f \to 1$), then according to equation (11) the efficiency will reduce to almost one, which means no improvement over the classical estimator V_{CC}. This is the case of the minimum efficiency. We comment that the situation of minimum efficiency usually does not occur in practice.

Having discussed the two extremes of the efficiency, it is natural to ask what the typical efficiency is when applied to real historical data, the answer of which relies on the quantity f when the number of periods n is fixed. From the classical statistics we know that the random variable $(1/f - 1)V_O/\mathrm{V}_C$ has an F distribution with both degrees being $n - 1$. This fact provides a method to estimate f from real historical data. The authors have tested about 2 dozen diverse U.S. equity historical daily data sets and found the average value of f over a long period to be between 0.18 and 0.30, depending on the underlying security. Thus, a typical value of f is about 0.25. Assuming we are interested in a biweekly variance estimation ($n = 10$), then the value of k_0 is 0.13 (from eq. [10] with $\alpha = 1.34$). Therefore, the typical efficiency for a biweekly variance estimation based on equation (7) is about 7.3, which means that in practice the variance of the new variance estimator is much smaller than the variance of the classical one based on closing prices only. Notice that the typical efficiency 7.3 is not too far from the maximum value 8.5 given by equation (13) when $n = 10$.

We know that in the case of zero drift, V_{GK} is the variance estimator with the minimum variance. Since the drift is usually small for daily stock price data, we investigate whether we have given up too much accuracy in exchange for drift independency by using V instead of V_{GK}. Let us compare the accuracy of our new variance estimator V with that of V_{GK} under the zero drift assumption. The variance of V_{GK} is $(2/n)[f^2 + 0.135(1 - f)^2]$, and the variance of V is $[2/(n - 1)][f^2 + k_0(1 - f)^2]$, where k_0 is given by equation (10) with $\alpha = 1.331$. Their ratio can be easily computed to be $\mathrm{Var}(V_{GK})/\mathrm{Var}(V) = 0.97[1 - 0.52/n + O(1/n^2)]$, where we have assumed f to be its typical value 0.25. For a 10-period estimate (assuming a 2-week daily data), using V_{GK} instead of V will only gain about 8% more accuracy. As the number of period n increases, the gain will gradually drop to only 3%. Therefore, we have shown that the drift independency of the variance estimator V is not gained at the expense of losing much accuracy.

When the drift is nonzero, V_{GK} has an upward bias, which depends on the parameters μ, σ, and T. The percentage of the bias is measured by the relative error $B = E[(V_{GK} - \sigma^2)/\sigma^2]$, where σ^2 is the true variance ($B = 0$ for our new estimator V, since it is unbiased for all drifts). V_{GK} has four components, namely V'_O, V'_C, V_P, and V_{RS} (see eq. [4]). Analytical expressions for the expectations of V'_O and V'_C can be determined easily for nonzero drifts (see app. A), and V_{RS} is independent of the drift. However, the expectation of V_P under nonzero drifts contains quite complicated expressions that involve multifold integrations and a summation. This complication is rooted from the complication in the joint probability density function of u and d. Although the final expression of the bias is quite complicated and difficult to evaluate numerically, the following qualitative properties still can be determined from our analysis: (i) the bias error B is a function of the dimensionless drift parameter $\mu\sqrt{T}/\sigma$ only; (ii) the percentage bias B is a monotonic function of the dimensionless drift parameter $\mu\sqrt{T}/\sigma$; (iii) the asymptotic behavior of the function is quadratic for large value of $\mu\sqrt{T}/\sigma$.

When we applied both V_{GK} and our drift independent estimator to daily stock price data, we found that the difference between the two estimations is usually negligible. This is due to the fact that the dimensionless drift parameter $\mu\sigma/\sqrt{T}$ is usually small (assuming using daily data). However, we do find periods in stock prices of some high-tech companies where the 10-day V_{GK} is approximately 20% larger than V when the stock undergoes a steady upward motion. This phenomenon of overestimation is commonly observed in the option markets, namely, the implied volatility drops dramatically (10%–20%) when the underlying stock is in a steady upward movement (downward movement in real life tends to be more violent). Our new variance estimator V correctly reflects this volatility drop, since a large component of V—V_{RS} is zero for a one-direction price movement, as we mentioned in the introduction.

We now suggest an extension of the variance estimator given by equation (7) to the situation where opening jumps exist but opening prices are not available. The situation arises in practice, since opening prices were not recorded for some U.S. equity historical data files dating back a few years. There is insufficient information to construct a highly efficient and mathematically consistent variance estimator in this case. However, a simple approximation can be made, in which we set the current period's opening price O_1 to the previous period's closing price C_0 (i.e., assuming no opening jumps). The key of the modification is to set $u = 0$ if $u < 0$, which is equivalent to set the current period's high to the larger of the current period's trading high and the previous period's closing price and similarly to set $d = 0$ if $d > 0$. The variance estimator given by equation (7) is then used after these modifications (the V_O component is always zero by construction). The result of this approximation tends to underestimate the true variance; that is, the estimator V will have a downward bias against the estimator V_{CC}, because the extremes of a single period obtained by this approximation underestimate the corresponding true extremes of the underlying continuous random walk.

III. Discussions on Discretization Error

The unbiasedness of variance estimators that use high and low information is only true under the continuous random walk limit. If one performs numerical simulations (finite step-size random walk), these estimators will have a downward bias. As the number of steps n within a single period approaches infinity (or the step size $h \equiv 1/N$ approaches zero), the downward bias disappears.

The fact that a variance estimator based on high and low prices will have a downward bias when step size is finite was noticed by both Garman and Klass (1980) and Rogers and Satchell (1991). The root of the bias is that the discretized maximum/minimum is always smaller/larger than the corresponding continuous one. If we denote Δ

as the absolute value of the discrepancy between the discretized and corresponding continuous extremes, then Rogers and Satchell (1991) showed that $E[\Delta] = a\sigma\sqrt{h}$ and $E[\Delta^2] = b\sigma^2 h$, where h is the step size and the constants a and b are given as $a = \sqrt{2\pi}[1/4 - (\sqrt{2} - 1)/6]$ and $b = (1 + 3\pi/4)/12$. Furthermore, a correction formula was derived by them, which for a single period is

$$\sigma_{RS}^2 = u(u - c) + d(d - c) + 2(u - d)a\sigma_{RS}\sqrt{h} + 2b\sigma^2 h, \tag{14}$$

where u, d, and c are high, low, and closing values, respectively, from a discretized simulation. The quadratic equation (14) is then solved to obtain the corrected volatility σ_{RS}, which agrees well with the true (continuous) volatility value for simulations of different step sizes. However, in practice the step size h (or the number of transactions) is in general not known; neither can h be inferred from other available data-like volume information. Therefore, the correction formula given by equation (14) cannot be applied directly in practice. It is desirable to have an estimator that uses high, low, and closing prices and is less sensitive to the step size h—namely, removing the leading order error term, which is proportional to $\sqrt{h}$. We now develop such an estimator.

The symmetry argument presented in appendix A is still valid for discretized data. Thus, the only quantities involving extremes that can be used to construct a variance estimator are combinations of V_P given by equation (2) and V_{RS} given by equation (3). The following combination,

$$\bar{V} = \frac{1}{2\ln 2 - 1}(2 \ln 2\ V_P - V_{RS}), \tag{15}$$

produces a variance estimator that is insensitive to h. In order to see this we perform an analysis similar to that of Rogers and Satchell and derive a single-period correction formula for this combined estimator

$$\begin{aligned}\bar{\sigma}^2 = {} & [(u - d)^2 - 2u(u - c) - 2d(d - c) \\ & - 2(b - a^2)\bar{\sigma}^2 h]/(4 \ln 2 - 2).\end{aligned} \tag{16}$$

It is clear that the leading order correction term in equation (14), which is proportional to $\sqrt{h}$, is absent from equation (16). Using the values of a and b given above, the coefficient of σ^2 in the correction term can be computed to be $0.2h$, which is very small since $h \equiv 1/\mathrm{N} \ll 1$. Therefore, the variance estimator equation (15) is indeed insensitive to the finite step size h. Notice that the variance estimator $\bar{V}$ is only valid when there is no drift ($\mu = 0$), since V_P is only valid under the zero drift condition. The V itself as a variance estimator is very inaccurate (assuming no opening jumps). It is not difficult to show that $\mathrm{Var}(\bar{V}) > \mathrm{Var}(V_C')$, where V_C' given by equation (6) is equivalent to the close-to-close variance estimator under the assumption of no opening jumps and zero drift.

The minimum-variance estimator under the restrictions of zero drift and finite step-size insensitivity is of the following linear combination:

$$V' = V'_O + k'V'_C + (1 - k')\bar{V}, \tag{17}$$

where V'_O and V'_C are defined by equations (5) and (6), respectively. The value of k' can be computed to be $k'_O \approx 2.6$. The variance of V' under this k' is $(2/n)[f^2 + 0.41(1 - f)^2]$. The variance ratio of V' in equation (17) to V in equation (7) under the typical value $f = 0.25$ is $\mathrm{Var}(V')/\mathrm{Var}(V) = 2.06[1 - 0.52/n + O(1/n^2)]$. Therefore, the variance of V' is about twice the variance of V for a moderate n. Thus, in gaining finite step-size insensitivity, we have given up the drift independency and a lot of accuracy.

IV. Conclusion

Given a historical data set containing n ($n > 1$) periods of high, low, open, and close prices, one should use the formula in equation (7) to compute the variance (volatility squared) of the underlying security during these periods, where the constant k is set to

$$k_0 = \frac{0.34}{1.34 + \dfrac{n + 1}{n - 1}}. \tag{18}$$

It is shown that the new estimator V given by equation (7) is the minimum-variance unbiased variance estimator, which is independent of both the drift and opening jumps of the underlying price movement. In practice, the result of V given by equation (7) in general will be much more accurate than the one given by the classical estimator V_{CC} given by equation (1), based on closing prices only.

Appendix A

Symmetries of a Variance Estimator

We now provide details of how the new variance estimator displayed in equation (7) is constructed. The key concept used here for a parameter estimator construction is from the work of Garman and Klass (1980; app. A), which stated that, if the joint probability density distribution of the observed data has certain measure-preserving symmetries, then the formula of the minimum-variance estimator for a parameter that remains unchanged under the symmetry transformations should be invariant under the same symmetry transformations. They proved that otherwise one could always symmetrize an estimator that does not have the full symmetry and obtain an improved estimator. We emphasize that it is not required for the symmetry transformations to leave all parameters unchanged—only the parameter being estimated needs to remain the same under the transformations.

A variance estimator should be a quadratic expression on the high, low, open, and close prices, which is due to the scale-invariant property of the joint density distribution and the analytic requirement in the neighborhood of the origin. The reason for the analytic requirement is that an estimator should be applicable to a constant time series (i.e., $u = d = c = o = 0$). In a quadratic expression, each term can only involve quantities from at most two different periods. Thus, we now look for a variance estimator for two data periods only. All the key steps in a multiperiod variance estimator construction are illustrated in the two-period one.

Let $\Theta(o_1, u_1, d_1, c_1, o_2, u_2, d_2, c_2)$ be any quadratic variance estimator based on the eight observables of the two periods. There will be a total of 36 coefficients for a general quadratic expression of eight variables. The symmetry of relabeling period 1 and period 2 immediately reduces the number of independent coefficients to 20. Using the following symmetries of the Brownian motion, the number of independent coefficients can be reduced much further. We refer readers to Garman and Klass (1980) for detailed discussions on Brownian motion symmetry transformations applied to single-period quantities. The symmetry transformations for quantities of two different periods are listed below. The first symmetry is the spatial reflection on both periods, under which Θ is transformed into $\Theta(-o_1, -d_1, -u_1, -c_1, -o_2, -d_2, -u_2, -c_2)$; the second symmetry is the time reversal on both periods, under which Θ is transformed into $\Theta(-o_1, u_1 - c_1, d_1 - c_1, -c_1, -o_2, u_2 - c_2, d_2 - c_2, -c_2)$; the third symmetry is the combination of the aforementioned two symmetries, that is, applying the spatial reflection to one period and the time reversal to the other period, under which Θ is transformed into $\Theta(-o_1, -d_1, -u_1, -c_1, -o_2, u_2 - c_2, d_2 - c_2, -c_2)$; the fourth symmetry is the exchanging of opening jumps, under which Θ is transformed into $\Theta(o_2, u_1, d_1, c_1, o_1, u_2, d_2, c_2)$. Notice that all four symmetry transformations leave the parameter σ unchanged, whereas μ is changed to $-\mu$ under the first three transformations. The requirement that the quadratic expression Θ be invariant under these four symmetry transformations reduces the number of independent coefficients from 20 to seven. After reparameterization, the quadratic expression Θ can be written as

$$\begin{aligned}\Theta(o_1, u_1, d_1, c_1, o_2, u_2, d_2, c_2) = \sum_{i=1}^{2} &[a_1(u_i - d_i)^2 + a_2(u_i^2 - u_i c_i + d_i^2 - d_i c_i) \\ &+ a_3 c_i^2 + a_4 o_i^2] + a_5(o_1 + o_2)(c_1 + c_2) \\ &+ a_6 c_1 c_2 + a_7 o_1 o_2,\end{aligned} \tag{A1}$$

where $a_1, \ldots, a_7$ are the independent coefficients.

We now require that Θ be an unbiased variance estimator, which means

$$E[\Theta] = \sigma^2. \tag{A2}$$

Notice that we require that equation (A2) be truly independent of the drift μ and the opening jump f. The joint probability density of u and d for a nonzero drift Brownian motion is given in Borodin and Salminen (1996); thus the quantity $E[(u - d)^2]$ can be computed by performing proper integrations. The result is a summation of a series of terms containing integrals. Further simplification of this result has not been found. A simple numerical calculation shows that the dependence of $E[(u - d)^2]$ on μ is more complicated than a quadratic function, hence a_1 must be set to zero. The other terms in equation (A1) have the following expectations: $E[u^2 - uc + d^2 - dc] = \sigma^2(1 - f)$, $E[c^2] = \sigma^2(1 - f) + \mu^2(1 - f)^2$, $E[o^2] = \sigma^2 f + \mu^2 f^2$, $E[(o_1 + o_2)(c_1 + c_2)] = 4\mu^2 f(1 - f)$, $E[c_1 c_2] = \mu^2(1 - f)^2$ and $E[o_1 o_2] = \mu^2 f^2$. Substituting all these expectations into equation (A2) and equating the corresponding coefficients produces the following one-parameter expression ($k \equiv 1 - a_2$)

$$\Theta = \frac{1}{2}(o_1 - o_2)^2 + k\frac{1}{2}(c_1 - c_2)^2 + (1 - k)\frac{1}{2}\sum_{i=1}^{2}[u_i(u_i - c_i) + d_i(d_i - c_i)], \tag{A3}$$

which is a special case of equation (7) when $n = 2$.

The generalization of the two-period variance estimator construction to that of the n period is straightforward. The resulting expression for the variance estimator is given in equation (7).

Appendix B

Zero Correlation between V_C and V_{RS}

In this appendix we prove that there is no correlation between V_C and V_{RS}. The proof consists of two steps. The first is to prove that within a single period of length t (equal to $[1 - f]T$) the quantities $(c - \mu t)^2$ and $u(u - c) + d(d - c)$ are uncorrelated; that is, $E[(c - \mu t)^2\,(u(u - c) + d(d - c))] = E[(c - \mu t)^2]\,E[u(u - c) + d(d - c)] = \sigma^4 t^2$. The proof relies on the probability density of the joint distribution of u and c (Borodin and Salminen 1996), which is

$$\rho(u, c) = \frac{2(2u - c)}{\sqrt{2\pi t^3}\sigma^3}\exp\left[\frac{(2u - c)^2}{2\sigma^2 t} + \frac{\mu}{\sigma^2}c - \frac{\mu^2 t}{2\sigma^2}\right]. \tag{B1}$$

Given the joint distribution of equation (B1), one can then compute $E[(c - \mu t)^2 u(u - c)]$, which gives the result $\frac{1}{2}\sigma^4 t^2$ (after some cumbersome algebra). The double integration is first carried out on the variable c with limits from $-\infty$ to u; it is then carried out on the variable u with limits from 0 to ∞. Similarly, we have $E[(c - \mu t)^2\, d(d - c)] = \frac{1}{2}\sigma^4 t^2$. Combining the two parts completes our first step of the proof. We comment here that the same brutal force method can be used to prove $E[u(u - c) + d(d - c)] = \sigma^2 t$, but the proof given by Rogers and Satchell (1991) using the Laplace transformation method is simpler.

The second step of our proof is to extend the single-period result to multiple periods. We first ‘‘detrend’’ all the c_i in V_C by defining a new $\tilde{c}_i$ to be $c_i - \mu t$, then expand the sums involved in the product $V_C V_{RS}$. The result $E[V_C V_{RS}] = E[V_C]\,E[V_{RS}]$ is then easily obtained, since the quantities involved in the cross terms from two different periods are uncorrelated, and the result given in the first step shows that the corresponding cross terms in the same period ($\tilde{c}_i^2$ and V_{RSi}) are also uncorrelated.

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