

Ordering of Risk Measures for Capital Adequacy

Julia L. Wirch	Mary R. Hardy
Dept. of Mathematics and Statistics	Dept. of Statistics and Actuarial Science
Simon Fraser University	University of Waterloo
jwirch@sfu.ca	mrhardy@uwaterloo.ca
phone: (604) 291-3340	phone: (519) 885-1211 x5510
fax: (604) 291-4947	fax: (519) 746-1875

Abstract

In this paper we show some features of distortion risk measure orders. We look at necessary and sufficient conditions for coherency, and for consistency with second order stochastic dominance. The results are related to current risk measures used in practice, such as value-at-risk (VaR) and the conditional tail expectation (CTE), also known as tail-VaR.

1 Introduction

1.1 Outline

In this section we review the coherence properties for a risk measure of Artzner *et al* (1999), and review risk measures based on distortion functions. In Section 2 we consider sufficient and necessary conditions for a distortion risk measure to be coherent. In Section 3 we consider consistency of the risk measure with second order stochastic dominance.

1.2 Coherent capital requirements

Given a loss random variable X , the risk measure is a functional $\rho(X) : X \mapsto [0, \infty)$. The premium principle is the most commonly recognized actuarial risk measure, but principles for capital requirements are also coming into recognition. Capital requirement risk measures are used to determine the capital required in respect of a random loss X with a view to avoiding insolvency. The most common capital requirement risk measure in common use is the Value at Risk, or VaR measure. This is better known by actuaries as a quantile reserving principle, where the reserve requirement is a quantile of the loss distribution.

Many of the accepted requisites for premium risk measures also apply to capital requirement risk measures. One of the major differences lies in recognition of gains. For a risk which may result in a gain ($X \leq 0$) or a loss ($X > 0$), it is appropriate in pricing the risk to take consideration of the potential gains. In capital adequacy this can lead to unacceptable results – for example, a negative capital requirement. For results consistent with the objectives of capital requirement calculations, it is appropriate to use a loss distribution censored at zero, and this censoring is assumed in this paper.

In Artzner *et al* (1999), a set of four axioms for a ‘coherent’ risk measure for capital adequacy are proposed. It is then demonstrated that a quantile risk measure, such as Value-at-Risk, or VaR, does not satisfy these axioms, but that

a measure based on the expectation in the right tail of the loss distribution, (Conditional Tail Expectation, or Tail-VaR) does satisfy the axioms and is therefore preferable.

Their coherency axioms for a risk measure are:

1. bounded above by the maximum loss ($\rho(X) \leq \max(X)$)
2. bounded below by the mean loss ($\rho(X) \geq E(X)$)
3. scalar additive and multiplicative ($\rho(aX + b) = a\rho(X) + b$)
4. subadditive ($\rho(X + Y) \leq \rho(X) + \rho(Y)$)

The conditional tail expectation (CTE) is defined for smooth distribution functions, given the parameter α , $0 < \alpha < 1$, as:

$$CTE_\alpha = E[X \mid X > F_X^{-1}(\alpha)].$$

where $F_X^{-1}()$ is the inverse distribution function of the loss random variable, X . That is, $F_X^{-1}(\alpha)$ is the 100α percentile of the loss distribution.

1.3 Distortion risk measures

Distortion functions and distortion risk measures developed from research on premium principles by Wang (1995), and are defined as follows:

A distortion function is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, and $g : [0, 1] \rightarrow [0, 1]$.

A distortion risk measure for a random loss X with decumulative distribution function $S(x)$ is

$$\rho(X) = \int_0^\infty g(S(x))dx$$

where $g()$ is the associated distortion function.

The distortion risk measure adjusts the true probability measure to give more weight to higher risk events, that is, $g(S(x))$ can be thought of as a risk adjusted decumulative distribution function. Since X is a non-negative random variable, $\rho(X) \equiv E_g[X]$ where the subscript indicates the change of measure.

Using distorted probabilities, it is possible to define a distortion $g_V()$ that will produce the traditional VaR measure, V_α , as the risk measure.

$$g_V(t) = \begin{cases} 1 & \text{if } 1 - \alpha < t \leq 1, \\ 0 & \text{if } 0 < t < 1 - \alpha. \end{cases} \quad (1)$$

so that the risk measure is

$$\rho_v(X) = \int_0^\infty g_v(S(x))dx = \int_0^{V_\alpha} dx = V_\alpha$$

where V_α is $F_X^{-1}(\alpha)$

The CTE or Tail-VaR can also be expressed in terms of a distortion risk measure as follows:

$$g_c(t) = \begin{cases} 1 & \text{if } 1 - \alpha < t \leq 1, \\ \frac{t}{1-\alpha} & \text{if } 0 < t < 1 - \alpha. \end{cases} \quad (2)$$

Both of these measures use only the tail of the distribution. Using the work of Wang (1995, 1996) and Wang, Young and Panjer (1997) on premium principles, it can be seen that there are advantages to distortions which utilize the whole censored loss distribution. The Beta-distortion risk measure uses the incomplete beta function:

$$g(S(x)) = \beta(a, b; S(x)) = \int_0^{S(x)} \frac{1}{\beta(a, b)} t^{a-1} (1-t)^{b-1} dt = F_\beta(S(x))$$

where $F_\beta(x)$ is the distribution function of the beta distribution, and $\beta(a, b)$ is the beta function with parameters $a > 0$ and $b > 0$; that is:

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

The Beta-distortion risk measure is concave if and only if $a \leq 1$ and $b \geq 1$, and is strictly concave if a and b are not both equal to 1 (see Wirch (1999b)).

The PH-transform is a special case of the Beta-distortion risk measure. The PH-transform risk measure is defined as:

$$\rho_{PH}(X) = \int_0^\infty S_X(x)^{\frac{1}{\gamma}} dx, \quad \gamma > 1 \tag{3}$$

2 Coherence and concave distortions

Theorem 2.1 (Wang, 1996). *If g is a concave distortion function, and $S_X(x) = 1 - F_X(x)$, then the distorted risk measure, $\rho_g(X)$ is a coherent risk measure.*

Proof: The first three coherency properties are proved for all concave distortion functions in Wang(1996). The fourth, sub-additivity, is stated but not proved in Wang (1996). We present the full proof which is closely based on that in Wang(1995), and is attributed there to Hesselager:

First note that for any concave distortion function $g(\cdot)$, for any $0 < a < b$ and $x > 0$,

$$g(b+x) - g(a+x) \leq g(b) - g(a). \tag{4}$$

For any arbitrary increasing concave distortion function g , we define $\rho_g(X) = \int_0^\infty g(S_X(x)) dx$. Using mathematical induction for every g and related ρ_g , we prove the result for arbitrary loss random variable V , and U a discrete loss random variable taking values in $\{0, \dots, n\}$. By

scalar additivity the proof also holds for $U \in \{k, \dots, n+k\}$ and by scalar multiplicativity for $U \in \{hk, \dots, (n+k)h\}$, $h > 0$. Any random variable can be approximated arbitrarily closely by a discrete variable with small span h .

By mathematical induction:

(i) For $n = 0, U_0 = 0$ almost surely, and $\rho(U_0) = 0$, so for any V

$$\rho(V + U_0) = \rho(V) + 0. \quad (5)$$

(ii) For $n, U_n \in \{0, \dots, n\}$, we assume that

$$\rho(U_n + V) \leq \rho(U_n) + \rho(V). \quad (6)$$

(iii) For $n + 1$: Consider (U_{n+1}, V) with $U_{n+1} \in \{0, 1, \dots, n + 1\}$, and let (U^*, V) be distributed as $(U_{n+1}, V | U_{n+1} > 0)$. By (ii) and scalar additivity the result holds for $U^* \in \{1, \dots, n + 1\}$. That is:

$$\rho(U^* + V) \leq \rho(U^*) + \rho(V). \quad (7)$$

With $\omega_0 = \Pr(U = 0)$ and $S_{V|0}(t) = \Pr(V > t | U = 0)$, we have for $t > 0$ that

$$S_U(t) = (1 - \omega_0)S_{U^*}(t),$$

$$S_V(t) = \omega_0 S_{V|0}(t) + (1 - \omega_0)S_V(t),$$

$$S_{U+V}(t) = \omega_0 S_{V|0}(t) + (1 - \omega_0)S_{U^*+V}(t).$$

This yields (according to Equation (4)) for $t > 0$,

$$\begin{aligned} g[S_{U+V}(t)] &= g[S_U(t)] - g[S_V(t)] \\ &= g[\omega_0 S_{V|0}(t) + (1 - \omega_0)S_{U^*+V}(t)] \\ &\quad - g[(1 - \omega_0)S_{U^*}(t)] - g[\omega_0 S_{V|0}(t) + (1 - \omega_0)S_V(t)] \\ &\leq g[(1 - \omega_0)S_{U^*+V}(t)] \\ &\quad - g[(1 - \omega_0)S_{U^*}(t)] - g[(1 - \omega_0)S_V(t)] \end{aligned}$$

$$= g(1 - \omega_0) \left\{ \frac{g[(1 - \omega_0)S_{U+V}(t)]}{g(1 - \omega_0)} - \frac{g[(1 - \omega_0)S_U(t)]}{g(1 - \omega_0)} - \frac{g[(1 - \omega_0)S_V(t)]}{g(1 - \omega_0)} \right\}$$

Now define $h[S(t)] = \frac{g[(1 - \omega_0)S(t)]}{g(1 - \omega_0)}$, a new increasing concave distortion function, and since $g(1 - \omega_0)$ is a positive constant, integration over t on both sides implies that the right hand side is less than zero by Equation (7), and this yields

$$\rho(U + V) \leq \rho(U) + \rho(V). \quad \square \tag{8}$$

So a sufficient condition for a coherent distortion risk measure is a concave distortion function. In fact, the first and third coherence properties are satisfied by any distortion; only the second and fourth use the concavity of $g(\cdot)$.

It is interesting also to investigate necessary conditions for coherence.

$$\text{The inequality } g(y) \geq y \quad \forall y \in [0, 1] \tag{9}$$

is satisfied for all concave distortion functions. This inequality is a necessary condition for a distortion risk measure to satisfy the second and fourth coherence properties; that is, that the risk measure is bounded below by the expected value of the loss, and that the risk measure is sub-additive.

Theorem 2.2 *Let g be a distortion function. If there exists some A such that for $t \in A$ $g(t) < t$ then the distorted risk measure, $\rho_g(X)$:*

- (a) *is not bounded below by $E[X]$ and*
- (b) *is not subadditive.*

Proof: (a) For a RV X , let $g(S(x)) < S(x)$ for some $a < X < b$;

Let

$$Y = \begin{cases} a & \text{if } X \leq a \\ X & \text{if } a < X \leq b \\ b & \text{if } X > b. \end{cases} \quad (10)$$

Then

$$S_Y(y) = \begin{cases} 1 & \text{if } y \leq a \\ S_X(y) & \text{if } a < y \leq b \\ 0 & \text{if } y > b. \end{cases} \quad (11)$$

Then

$$\begin{aligned} E[Y] &= a + \int_a^b S_X(y) dy \\ &< a + \int_a^b g(S_X(y)) dy \\ &< \rho(Y) \end{aligned}$$

(b) Let $A \subset [0, 1]$ such that $\forall t \in A, g(t) < t$. Then

$$\exists c \in A \text{ such that } \forall t \in [0, 1], c - g(c) \geq t - g(t) \quad (12)$$

Assume first that $c \leq 0.5$, so that

$$c - g(c) \geq 2c - g(2c) \Rightarrow g(2c) \geq c + g(c) \quad (13)$$

Let $U \sim \text{Uniform}[0, 1]$, and define Y and Z

$$Y = \begin{cases} 1 & \text{if } U \leq c \\ 0 & \text{if } U > c. \end{cases}$$

and

$$Z = \begin{cases} 0 & \text{if } U \leq 1 - c \\ 1 & \text{if } U > 1 - c. \end{cases}$$

so that

$$Y + Z = \begin{cases} 0 & \text{if } c < U \leq 1 - c \\ 1 & \text{if } U \leq c \text{ or } U > 1 - c. \end{cases}$$

then $\rho(Y) = \rho(Z) = g(c)$ and $\rho(Y + Z) = g(2c)$

so that $\rho(Y + Z) = g(2c) \geq c + g(c)$ from (13), and, since $c \in A$, $c > g(c)$ and $g(2c) > 2g(c)$, that is $\rho(Y + Z) > \rho(Y) + \rho(Z)$ as required.

Now let $c > 0.5$, and define Y, Z as above. Now

$$Y + Z = \begin{cases} 2 & \text{if } 1 - c < U \leq c \\ 1 & \text{if } U \leq 1 - c \text{ or } U > c. \end{cases}$$

then $\rho(Y + Z) = 1 + g(2c - 1)$. From (12) $g(2c - 1) \geq c - 1 + g(c)$, and, since $c \in A$, $c - 1 + g(c) > 2g(c) - 1$.

So $\rho(Y + Z) = 1 + g(2c - 1) > 1 + 2g(c) - 1 = \rho(Y) + \rho(Z)$ as required.

□

The VaR distortion does not satisfy the inequality (9), as, for parameter α , $g(t) = 0 < t$ for $t < 1 - \alpha$ (see equation (1)). The VaR measure therefore is not coherent. Examples of the problems with VaR are given in Wirch(1999a) and Wirch and Hardy(2000).

Both the CTE and beta distortion functions are concave and therefore coherent. However, the beta distortion function is strictly concave ($g''_{\beta}(t) < 0$), whereas the piecewise linear distortion functions, including the CTE, are not ($g''_{\beta}(t) = 0$). This makes a difference in the following section where we consider stochastic ordering.

3 Stochastic Order

We say that a risk measure $\rho(\cdot)$ preserves a stochastic ordering \prec if

$$X \prec Y \implies \rho(X) \leq \rho(Y).$$

The ordering is strongly preserved if strict stochastic ordering of random variables leads to strict ordering of the risk measure.

First Order Stochastic Dominance If $S_X(t) \leq S_Y(t)$ for all $t \geq 0$, and $S_X(t) < S_Y(t)$ for some $t \geq 0$, then $X \prec_{1st} Y$.

(Note: There are many other equivalent conditions. See Wang (1998).)

All distortion functions preserve first order stochastic dominance. That is, if $X \prec_{1st} Y$ then $\rho(X) \leq \rho(Y)$ follows from the fact that the distortion function is an increasing function.

Second Order Stochastic Dominance : For any two risks X and Y , if

$$\int_x^\infty S_X(t)dt \leq \int_x^\infty S_Y(t)dt,$$

for all $x \geq 0$, with strict inequality for some $x \in (0, \infty)$ then we say that X precedes Y in second order stochastic dominance, or $X \prec_{2nd} Y$.

(Note: There are many other equivalent conditions such as stop-loss order. See Wang(1998))

Not all risks can be ordered using second order stochastic dominance. Any pair of risks with survival distributions which cross an even number of times, cannot be compared, as the sign of the difference in integrals before the first crossing and after the last crossing are opposite.

Where we can order random variables with second order stochastic dominance, an additional property that would be attractive in a risk measure is that it strongly preserves second order stochastic dominance. In fact, as we show below, this depends on whether or not the distortion function is strictly concave.

Theorem 3.1 For a risk measure $\rho(X) = \int_0^\infty g(S(x))dx$ where $g()$ is strictly concave, then $X \prec_{2nd} Y \Rightarrow \rho(X) < \rho(Y)$

Proof: (based on proof from Wang (1996))

Due to Müller (1996) we only have to prove that the increasing, strictly concave distortion risk measures preserve second order stochastic dominance where the decumulative distribution functions cross once only.

Let $E[X] \leq E[Y]$, $X \prec_{2nd} Y$ and let t_0 be the once crossing point, so that

$$\begin{aligned} S_X(t) &\geq S_Y(t) \quad \text{for } t < t_0 \\ S_X(t) &\leq S_Y(t) \quad \text{for } t \geq t_0 \end{aligned}$$

and since $X \prec_{2nd} Y$ either

$$S_X(t) < S_Y(t) \quad \text{for some } t > t_0$$

$$\text{and/or } S_X(t) > S_Y(t) \quad \text{for some } t < t_0$$

Next, construct a new ddf,

$$S_Z(t) = \max\{S_X(t), S_Y(t)\} = \begin{cases} S_X(t) & t < t_0 \\ S_Y(t) & t \geq t_0 \end{cases}$$

so that:

$$\rho_g(Z) - \rho_g(X) \geq \frac{g[S_X(t_0)]}{S_X(t_0)} \int_{t_0}^{\infty} [S_Y(t) - S_X(t)] dt \quad (14)$$

and

$$\rho_g(Z) - \rho_g(Y) \leq \frac{g[S_X(t_0)]}{S_X(t_0)} \int_0^{t_0} [S_X(t) - S_Y(t)] dt \quad (15)$$

with at least one of the above inequalities being a strict inequality.

Subtracting the last two equations, we obtain

$$\rho_g(Y) - \rho_g(X) > \frac{g[S_X(t_0)]}{S_X(t_0)} \int_0^{\infty} [S_Y(t) - S_X(t)] dt \geq 0 \quad (16)$$

Thus, $\rho_g(Y) > \rho_g(X)$. \square

If the distortion function $g(t)$ is not strictly concave, that is $g''(t) = 0$ for some t , then the risk measure preserves second order stochastic dominance only weakly, that is we can find Y such that $X \prec_{2nd} Y$ but $\rho(X) = \rho(Y)$

Theorem 3.2 *A risk measure derived from a distortion function which is concave but not strictly concave does not strongly preserve second order stochastic dominance.*

Proof: The proof parallels the proof of Theorem 3.1; however over any linear portion of the distortion function, we have $\frac{g[S_X(t)]}{S_X(t)} = M$, a constant, the slope of the linear portion.

For any risk X we can construct a risk Y such that $E[X] = E[Y]$ and $X \prec_{2nd} Y$, and where t_0 , the once crossing point, is such that $S_X(t_0) = S_Y(t_0) = b$, and $g(b)$ lies on one linear portion of the distortion function. Also suppose that the linear portion containing $g(b)$ covers the range from $g(a)$ to $g(c)$ where $a < b < c$. Then

$$S_Y(t) \begin{cases} = S_X(t) & \text{for } t \geq t_a \\ = a & \text{for } t = t_a \\ \leq S_X(t) & \text{for } t_a \geq t \geq t_b \\ = b & \text{for } t = t_b \\ \geq S_X(t) & \text{for } t_b \geq t \geq t_c \\ = S_X(t) & \text{for } t \leq t_c \end{cases} \quad (17)$$

which implies that

$$\int_0^{t_c} [S_Y(t) - S_X(t)] dt = 0 \quad (18)$$

and

$$\int_{t_a}^{\infty} [S_Y(t) - S_X(t)] dt = 0. \quad (19)$$

Constructing the same inequalities as in (14), (15), (16), we obtain

$$\rho_g(Z) - \rho_g(X) = \frac{g[S_X(t_0)]}{S_X(t_0)} \int_{t_b}^{t_a} [S_Y(t) - S_X(t)] dt \quad (20)$$

and

$$\rho_g(Z) - \rho_g(Y) = \frac{g[S_X(t_0)]}{S_X(t_0)} \int_{t_c}^{t_b} [S_X(t) - S_Y(t)] dt \quad (21)$$

which gives,

$$\rho_g(Y) = \rho_g(X). \quad (22)$$

That is, we construct a risk such that $X \prec_{2nd} Y$ but $\rho_g(Y) = \rho_g(X)$. \square

3.1 Example

A simple example will illustrate the point of this section. Consider two random variables, X and Y :

$$X = \begin{cases} 0 & \text{with probability } 0.95 \\ 50 & \text{with probability } 0.025 \\ 100 & \text{with probability } 0.025. \end{cases}$$

$$Y = \begin{cases} 50 & \text{with probability } 0.975 \\ 100 & \text{with probability } 0.025. \end{cases}$$

Clearly Y has the riskier distribution, and clearly $X \prec_{2nd} Y$.

The CTE with parameter 95% is the expected value of the loss, given the loss lies in the upper 5% of the distribution. In both these cases, the CTE(95%) risk measure is 75. In fact, for any parameter $\alpha \geq 0.95$ the CTE of these two risks will be equal. The CTE risk measure cannot distinguish between these risks in general.

The beta distortion on the other hand will strictly order the risks, for any parameters $0 < a \leq 1$ and $b/geq 1$, provided aa and b are not both equal to 1. For example, let $b = 1$ and $a = 0.1$ (this gives the PH-transform risk measure). Then

$$\rho_\beta(X) = 50(.05)^{0.1} + 50(.025)^{0.1} = 71.63$$

and $\rho_\beta(Y) = 50 + 50(.025)^{0.1} = 84.58$.

4 Conclusions

In this paper, we have shown that the CTE and other partially linear distorted risk measures do not strongly preserve second order stochastic dominance. The beta-distortion risk measure and other risk measures with increasing, strictly concave distortion functions do preserve second order stochastic dominance, and are superior in ordering risk.

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