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Optimal inventory policy with fixed and proportional transaction costs under a risk constraint

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ABSTRACT

The traditional inventory models focus on characterizing replenishment policies in order to maximize the total expected profit or to minimize the expected total cost over a planned horizon. However, for many companies, total inventory costs could be accounting for a fairly large amount of invested capital. In particular, raw material inventories should be viewed as a type of invested asset for a manufacturer with suitable risk control. This paper is intended to provide this perspective on inventory management that treats inventory problems within a wider context of financial risk management. In view of this, the optimal inventory problem under a VaR constraint is studied. The financial portfolio theory has been used to model the dynamics of inventories. A diverse portfolio consists of raw material inventories, which involve market risk because of price fluctuations as well as a risk-free bank account. The value-at-risk measure is applied thereto to control the inventory portfolio's risk. The objective function is to maximize the utility of total portfolio value. In this model, the ordering cost is assumed to be fixed and the selling cost is proportional to the value. The inventory control problem is thus formulated as a continuous stochastic optimal control problem with fixed and proportional transaction costs under a continuous value-at-risk (VaR) constraint. The optimal inventory policies are derived by using stochastic optimal control theory and the optimal inventory level is reviewed and adjusted continuously. A numerical algorithm is proposed and the results illustrate how the raw material price, inventory level and VaR constraint are interrelated.

1. Introduction

Inventories are stocks of raw materials, components and finished goods that are stored in warehouses, the instrumentalities of transportation, and retail stores. Raw material inventories are necessary to manufacturers because they create buffers against irregular supplies and demand shifts, guarantying product availability. Yet, according to Ballou [1], stockpiling inventory may result in costs in the range of 20%–40% of annual invested capital. Thus, good inventory control will provide lower costs and promote overall company performance. Recognizing the importance of inventory management, copious literature investigating optimal inventory strategies exists. Arrow et al. [2] laid the foundation of modern inventory theory, in which expected costs are chosen as an objective function. The traditional inventory models focus on characterizing replenishment policies in order to maximize the total expected profit or to minimize the expected total cost over a planned horizon. Examples include the popular (EOQ) model and the (s, S) model [3,1,4]. The conventional inventory control methods are appropriate for risk-neutral managers, but in reality most managers are risk averse [5], and in the field of the supply chain,

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risk analysis and risk control have become more and more important [6,7]. Therefore, techniques that consider both risks and returns of holding inventory are crucial.

Risk management is increasingly important in many disciplines, ranging from banking [8] and technology [9–12] to humanity and ecology [13,14]. The literature on inventory models with risk control is quite limited and is mainly focused on discrete-time problems. Eeckhoudt et al. [15] investigate the effects of risk and risk aversion in the single-period inventory ("newsvendor") problem Agrawal and Seshadri [16] consider the single-period inventory problem for a risk-averse retailer, whose objective is to maximize expected utility. Bouakiz and Sobel [17] optimize the multi-period news vendor model with an exponential utility criterion. Chen et al. [18] extend the multi-period inventory model with a general risk-averse utility function and illustrate with numerical results the effects of risk aversion on inventory strategies. In recent years, the VaR measure has also received attention in the supply chain [19]. Some literatures of inventory management have also used VaR as a risk control measure. Luciano et al. [20] consider a standard multi-period static inventory model and define the optimal policy as one that maximizes the expected discounted profits. The mean and variance of profits (and costs) are obtained for both finite and infinite cycles. The VaR induced by the optimal replenishment policy is estimated by applying probability theory Tapiero [21] attempts to formulate a single period inventory model which is to minimize the VaR of the total cost. So far, the models considered are discrete in nature with fixed cycle length between replenishments. Clearly, sophisticated stochastic models have not been used to model inventory price fluctuation and continuous replenishment policy has not been studied.

The recent violent fluctuations in commodity prices have forced many manufacturers to endure large market risk. Under these circumstances material inventories not only ensure production availability, but are also a kind of investment asset. Their price fluctuation nature closely resembles the risky assets of a financial portfolio. Undoubtedly, the concept of portfolio theory has been well developed in finance. In this paper, the financial portfolio theory will be applied to address the inventory control problem.

The specific focus herein will be the industrial manufacturer's raw material inventory. It is assumed that the manufacturer can sell the raw material inventories back to the suppliers at a discounted market price. And once a non-zero order is placed, the manufacturer has to pay a fixed ordering cost to the supplier except purchasing costs. The problems are then addressed from the perspective of financial portfolio theory with fixed and proportional transaction costs. The raw material inventory required by a manufacturer is considered as an investment, and a portfolio consisting of these material inventories as well as a risk free bank account is explored. To exercise proper risk control over the inventory portfolio value, the VaR constraint is imposed continuously over time. The objective function is to maximize the total expected utility of the portfolio value during the horizon. The optimal ordering and selling conditions are derived by using stochastic optimal control theory. Under this formulation, the optimal inventory level can be reviewed and adjusted continuously. By applying the VaR constraint, and assuming that portfolio allocations do not change over a short horizon period, we indicate that holding value in the raw material inventory is reduced whenever the VaR constraint becomes active.

The rest of the paper is organized as follows. In Section 2, the model for the continuous-time optimal inventory portfolio without VaR constraint is derived for one raw material inventory plus a risk-free asset. After that, the VaR constraint is imposed in Section 3 and the optimal ordering and selling conditions are derived. The final optimal inventory policy under a VaR constraint is continuously reviewed and adjusted. Finally, in Section 4, a numerical algorithm is proposed to solve the optimal control problem and illustration examples are presented in Section 5.

2. Inventory model without VaR constraint

2.1. Model formulation

Consider an industrial plant producing only one kind of product and requiring only one type of raw material. Our focus is on the inventory policies of the raw material. Because of fluctuations in the price of the raw material, holding onto an inventory is a kind of risky asset. Given a sufficient amount of capital to maneuver, the manager has two choices: to invest in a risk free asset, such as bank notes, or to maintain raw material inventories. From the perspective of finance, the manager has a portfolio consisting of one risk free asset and one risky asset. For simplicity, the following assumptions are made:

- In the planned time horizon, the price of product is a deterministic function in respect to time t, which is exogenously determined by markets.
- The demand of product and the price of raw material are independent to each other.
- There is no lead time for the raw material.

First, some notations are introduced:

- $\omega_0(t)$ is the risk free asset value at time t.
- $\omega_1(t)$ is the holding value of raw material inventory at time t.
- $X(t)$ is the total value of the risk free asset plus the raw material inventory, that is

$$X(t) = \omega_0(t) + \omega_1(t).$$

- $S_0(t)$ is the risk free asset price at time t.
- $S_1(t)$ is the raw material price at time t.

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- $D(t)$ is the demand quantity for the product per unit time at time t.
- $P(t)$ is the product price at time t.
- $\bar{\omega}$ is the inventory quantity to be consumed for the raw material to produce one piece of product.
- ζ is the holding cost per unit value of the raw material per unit time.
- η is the transaction cost per unit value for selling the raw material.
- F is the fixed order cost on each non-zero order of the raw material.
- $\omega_0(t+)$ is the risk free asset value after rebalancing takes place at time t in the state of $(\omega_0(t), \omega_1(t))$.
- $\omega_1(t+)$ is the inventory value of the raw material after rebalancing takes place at time t in the state of $(\omega_0(t), \omega_1(t))$.

Next, the following dynamics are assumed:

(1) $D(t)$ is the demand rate of the product at time t. It follows that

$$D(t) = \mu_D + \sigma_D \tilde{B}(t), \tag{1}$$

where $\tilde{B}(t)$ is a standard Brownian motion, both μ_D and σ_D are constants. Thus the demand rate $D(t)$ follows a normal distribution as

$$D(t) \sim N\left(\mu_D, \sigma_D\sqrt{t}\right).$$

(2) $S_0(t)$ is a deterministic process at a deterministic short rate of interest r, which can be written as

$$dS_0(t) = rS_0(t)dt. \tag{2}$$

(3) $S_1(t)$ is the price process of the raw material, which is assumed to follow a geometric Brownian motion as

$$dS_1(t) = S_1(t)\left(\mu_S dt + \sigma_S dB(t)\right), \tag{3}$$

where $B(t)$ is a standard Brownian motion, μ_S and σ_S are constants. Solving the above SDE yields

$$S_1(t) = S_1(0)\exp\left\{\left(\mu_S - \frac{1}{2}{\sigma_S}^2\right)t + \sigma_S B(t)\right\}, \tag{4}$$

which implies the distribution of $S_1(t)$ is lognormal, that is

$$\ln S_1(t) \sim N\left(\alpha(t), \beta(t)\right),$$

where $\alpha(t) = \ln S_1(0) + \left(\mu_S - \frac{1}{2}{\sigma_S}^2\right)t$ and $\beta(t) = \sigma_S\sqrt{t}$.

Here, μ_S is the average return rate of the raw material inventory and since the raw material is a type of investment for the manufacturer, it is reasonable to assume the following condition

$$\mu_S - \zeta - r > 0. \tag{5}$$

For a small value of Δt, if there is no transaction and no consumption of the raw material between $(t, t + \Delta t)$, the risk free asset and raw material inventory holdings in $(t, t + \Delta t)$ evolve as the prices of the underlying assets change. According to the dynamical equations (2) and (3), we have

$$\omega_0(t + \Delta t) = \omega_0(t) + r\omega_0(t)\Delta t, \tag{6}$$

$$\omega_1(t + \Delta t) = \omega_1(t) + \omega_1(t)\left(\mu_S\Delta t + \sigma_S\Delta W(t)\right). \tag{7}$$

For any t, $0 \le t < T$, we define

(a) $q_B(t) = q_B(\omega_0, \omega_1, t) \ge 0$ is the capital value used to order raw material at time t in the state of (ω_0, ω_1).
(b) $q_S(t) = q_S(\omega_0, \omega_1, t) \ge 0$ is the inventory value of the raw material to be sold at time t in the state of (ω_0, ω_1).

Since there is no lead time for ordering and selling the material, the rebalancing takes place instantly. Then we have

$$\omega_0(t+) = \omega_0(t) + (1 - \eta)q_S(t) - q_B(t) - F\Theta\left[q_B(t)\right], \tag{8}$$

$$\omega_1(t+) = \omega_1(t) + q_B(t) - q_S(t), \tag{9}$$

where

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{10}$$

In our formulation, short-selling is not allowed for the raw material inventory, which provides the condition that

$$\omega_1(t) \ge 0. \tag{11}$$

And the assumption that raw material inventory will always fulfill production demand implies that

$$\omega_1(t+) \ge S_1(t)D(t)\bar{\omega}\Delta t. \tag{12}$$

According to Eqs. (6) and (7), the holdings of the assets at time $t+\Delta t$ can be written as

$$\begin{aligned}\omega_0(t+\Delta t) &= \omega_0(t+) + \Delta\omega_0(t+) + P(t)D(t)\Delta t - \zeta\omega_1(t+)\Delta t \\ &= \omega_0(t+) + r\{\omega_0(t) + (1-\eta)q_S(t) - q_B(t) - F\Theta[q_B(t)]\}\Delta t + (P(t)D(t) - \zeta\omega_1(t+))\Delta t,\end{aligned} \tag{13}$$

$$\begin{aligned}\omega_1(t+\Delta t) &= \omega_1(t+) + \Delta\omega_1(t+) - S_1(t)\bar{\omega}D(t)\Delta t \\ &= \omega_1(t+) + (\omega_1(t) + q_B(t) - q_S(t))\mu_S\Delta t - S_1(t)\bar{\omega}D(t)\Delta t\,(\omega_1(t) + q_B(t) - q_S(t))\sigma_S\Delta B(t).\end{aligned} \tag{14}$$

$X(t)$ is the total value of the risk free asset and the inventory at time t, that is

$$X(t) = \omega_0(t) + \omega_1(t). \tag{15}$$

Given a small time interval Δt, we define

$$\Delta X(t) = X(t+\Delta t) - X(t+). \tag{16}$$

Substituting Eqs. (13) and (14) into (16), we have

$$\begin{aligned}\Delta X(t) &= \omega_0(t+\Delta t) + \omega_1(t+\Delta t) - [\omega_0(t+) + \omega_1(t+)] \\ &= [P(t) - S_1(t)\bar{\omega}]D(t)\Delta t + r\{\omega_0(t) + (1-\eta)q_S(t) - q_B(t) - F\Theta[q_B(t)]\}\Delta t \\ &\quad + [\omega_1(t) + q_B(t) - q_S(t)][(\mu_S - \zeta)\Delta t + \sigma_S\Delta B(t)].\end{aligned}$$

Taking the limit as $\Delta t \to 0$, $0 \le t < T$, we get

$$\begin{aligned}dX(t) &= r\{\omega_0(t) + (1-\eta)q_S(t) - q_B(t) - F\Theta[q_B(t)]\}dt + [\omega_1(t) + q_B(t) - q_S(t)] \\ &\quad \times (\mu_S - \zeta)dt + [P(t) - S_1(t)\bar{\omega}]D(t)dt + [\omega_1(t) + q_B(t) - q_S(t)]\sigma_S dB(t),\end{aligned} \tag{17}$$

and

$$\omega_1(t+) = \omega_1(t) + q_B(t) - q_S(t) > 0. \tag{18}$$

The objective is to maximize the total expected value of the utility function, thus the optimal problem is

$$\sup_{q_B, q_S} E\left[\int_0^T U(t, X(t))\,dt + W(T, X(T))\right], \tag{19}$$

subject to

$$\begin{aligned}dX(t) &= r\{\omega_0(t) + (1-\eta)q_S(t) - q_B(t) - F\Theta[q_B(t)]\}dt + [\omega_1(t) + q_B(t) - q_S(t)] \\ &\quad \times (\mu_S - \zeta)dt + [P(t) - S_1(t)\bar{\omega}]D(t)dt + [\omega_1(t) + q_B(t) - q_S(t)]\sigma_S dB(t),\end{aligned} \tag{20}$$

$$\omega_1(t+) = \omega_1(t) + q_B(t) - q_S(t) > 0. \tag{21}$$

2.2. Derivation of the Hamilton–Jacobi–Bellman equation

In this section, we will derive the Hamilton–Jacobi–Bellman partial equation of the optimal control problem. The method we used is similar with the dynamic portfolio selection problem without transaction cost (see, for example, [22]). Define the optimal value function as

$$V(x,t) = \sup_{q_B, q_S} E_t\left[\int_t^T U(x,s)\,ds + W(T, X(T))\right], \tag{22}$$

where $x = \omega_0(t) + \omega_1(t)$ is a possible state of $X(t)$.
Then we assume the following:

(a) An optimal control $(\hat{q}_B, \hat{q}_S)$ exists;
(b) The optimal value function V is regular in the sense that $V \in C^{1,2}$;
(c) During rebalancing, the value of the value function is preserved [23], that is

$$V(x(t+), t) = V(x,t). \tag{23}$$

Note

$$G(q_B, q_S) \equiv r\{\omega_0 + (1-\eta)q_S - q_B - F\Theta(q_B)\} + (\omega_1 + q_B - q_S)(\mu_S - \zeta) + (P - s_1\bar{\omega})D,$$

and

$$H(q_B, q_S) = {\sigma_S}^2(\omega_1 + q_B - q_S)^2.$$

Given a small time interval Δt, expand the value function $V\left[x(t+\Delta t), t+\Delta t\right]$ at time t after rebalancing, that is at the point $(x(t+), t+)$, by Taylor's series. Using the properties of Brownian motion yields

$$E_t\left[V\left(x\left(t+\Delta t\right), t+\Delta t\right)\right] = V\left(x\left(t+\right), t+\right) + \frac{\partial V}{\partial t}(x,t)\,\Delta t + G\left(q_B, q_S\right)\frac{\partial V}{\partial x}(x,t)\,\Delta t + \frac{1}{2}H\left(q_B, q_S\right)\frac{\partial^2 V}{\partial x^2}(x,t)\,\Delta t + o\left(\Delta t\right),$$

where $o(\Delta t)$ refers to the truncated higher order terms.
According to the preservation property of the value function stated by Eq. (23), the Taylor expansion can be written as

$$E_t\left[V\left(x\left(t+\Delta t\right), t+\Delta t\right)\right] = V\left(x, t\right) + \frac{\partial V}{\partial t}(x,t)\,\Delta t + G\left(q_B, q_S\right)\frac{\partial V}{\partial x}(x,t)\,\Delta t + \frac{1}{2}H\left(q_B, q_S\right)\frac{\partial^2 V}{\partial x^2}(x,t)\,\Delta t + o\left(\Delta t\right). \tag{24}$$

Given the state point $(x \cdot t)$, consider the following two strategies over the interval $[t, T]$:
Strategy I. Use the optimal control $\left(\hat{q}_B, \hat{q}_S\right)$.
Strategy II. Use the control $({q_B}^*, {q_S}^*)$ defined as

$$\left({q_B}^*, {q_S}^*\right) = \begin{cases} \left(q_B(y,s), q_S(y,s)\right), & (y,s) \in R \times [t, t+\Delta t], \\ \left(\hat{q}_B(y,s), \hat{q}_S(y,s)\right), & (y,s) \in R \times (t+\Delta t, T]. \end{cases} \tag{25}$$

Expected utility for Strategy I. This is trivial, since by definition the utility is the optimal one given by $V(x,t)$.
Expected utility for Strategy II. Divide the time internal $[t, T]$ into two parts, the intervals of $[t, t+\Delta t]$ and $(t+\Delta t, T]$, respectively.

- The expected utility, using Strategy II, for the interval $[t, t+\Delta t]$ is given by

$$E_t\left[\int_t^{t+\Delta t} U(x,s)\,ds\right]. \tag{26}$$

- By the definition of the control $({q_B}^*, {q_S}^*)$, use the optimal strategy during the entire interval $[t+\Delta t, T]$. Thus the remaining expected utility at time $t+\Delta t$ is given by $V\left(x\left(t+\Delta t\right), t+\Delta t\right)$. Then the conditional expected utility over the interval $[t+\Delta t, T]$ on the state (x,t), is given by

$$E_t\left[V\left(x\left(t+\Delta t\right), t+\Delta t\right)\right]. \tag{27}$$

So the total expected utility for Strategy II is

$$E_t\left[\int_t^{t+\Delta t} U(x,s)\,ds + V\left(x\left(t+\Delta t\right), t+\Delta t\right)\right]. \tag{28}$$

Comparing these two strategies yields the following inequality because Strategy I is, by definition, the optimal one

$$V(x,t) \geq E_t\left[\int_t^{t+\Delta t} U(x,s)\,ds + V\left(x\left(t+\Delta t\right), t+\Delta t\right)\right]. \tag{29}$$

Substituting the Eq. (24) into the above inequality gives

$$\left[\frac{\partial V}{\partial t}(x,t) + G\left(q_B, q_S\right)\frac{\partial V}{\partial x}(x,t) + \frac{1}{2}H\left(q_B, q_S\right)\frac{\partial^2 V}{\partial x^2}(x,t)\right]\Delta t + E_t\left[\int_t^{t+\Delta t} U(x,s)\,ds\right] + o\left(\Delta t\right) \leq 0.$$

Dividing by Δt and letting Δt tend to zero provides

$$U(x,t) + \frac{\partial V}{\partial t}(x,t) + G\left(q_B, q_S\right)\frac{\partial V}{\partial x}(x,t) + \frac{1}{2}H\left(q_B, q_S\right)\frac{\partial^2 V}{\partial x^2}(x,t) \leq 0. \tag{30}$$

Since the control law (q_B, q_S) is arbitrary, this inequality will hold for all choices and we will have equality if and only if $(q_B, q_S) = \left(\hat{q}_B, \hat{q}_S\right)$. Thus we conclude that $V(x,t)$ satisfies the HJB-equation

$$\frac{\partial V}{\partial t}(x,t) + \sup_{q_B, q_S}\left(U(x,t) + G\left(q_B, q_S\right)\frac{\partial V}{\partial x} + \frac{1}{2}H\left(q_B, q_S\right)\frac{\partial^2 V}{\partial x^2}\right) = 0, \tag{31}$$

with the boundary condition

$$V(x,T) = 0, \qquad V(0,t) = 0. \tag{32}$$

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2.3. Establishment of optimal inventory policy

We know as a matter of common sense that it is not optimal to order and sell raw material simultaneously, so

$$q_B \times q_S = 0. \tag{33}$$

Then at time t in the state of x, the manager has three options: (i) order raw materials for inventory, (ii) sell the raw materials currently in inventory, or (iii) do nothing. So the total control set Ω can be split into three independent regions: (i) the ordering region Ω_B, (ii) the selling region Ω_S and (iii) the no transaction region Ω_{NT}, such that

$$\begin{aligned} &\Omega_B = \{(q_B, q_S = 0)\,|q_B > 0\}, \qquad \Omega_S = \{(q_B = 0, q_S)\,|q_S > 0\}, \\ &\Omega_{NT} = \{(q_B = 0, q_S = 0)\}, \quad \text{and} \quad \Omega = \Omega_B \cup \Omega_S \cup \Omega_{NT}. \end{aligned} \tag{34}$$

According to the HJB equation, the optimal problem is reduced to solve the static optimization problem provided by

$$\sup_{q_B, q_S} \left(U(x,t) + G(q_B, q_S) \frac{\partial V}{\partial x} + \frac{1}{2} H(q_B, q_S) \frac{\partial^2 V}{\partial x^2} \right).$$

2.3.1. The optimal ordering policy

In the ordering region $\Omega_B = \{(q_B, q_S = 0)\,|q_B > 0\}$, the static optimization problem is reduced to

$$\sup_{q_B > 0} \left(U(x,t) + G(q_B, 0) \frac{\partial V}{\partial x} + \frac{1}{2} H(q_B, 0) \frac{\partial^2 V}{\partial x^2} \right), \tag{35}$$

where

$$G(q_B, 0) \equiv r\{\omega_0 - q_B - F\} + (\omega_1 + q_B)(\mu_S - \zeta) + (P - s_1\bar{\omega})D,$$

and

$$H(q_B, 0) = {\sigma_S}^2 (\omega_1 + q_B)^2.$$

The first-order necessary conditions of the static optimization problem with respect to q_B are given by

$$(\mu_S - \zeta - r) \frac{\partial V}{\partial x} + {\sigma_S}^2 (\omega_1 + q_B) \frac{\partial^2 V}{\partial x^2} = 0, \tag{36}$$

and

$$\frac{\partial^2 V}{\partial x^2} < 0. \tag{37}$$

Rearranging (36) yields

$$q_{B\text{opt}}(x,t) = \frac{-V_x(\mu_S - \zeta - r)}{{\sigma_S}^2 V_{xx}} - \omega_1(x,t), \tag{38}$$

where $V_x = \frac{\partial V}{\partial x}$ and $V_{xx} = \frac{\partial^2 V}{\partial x^2}$.
In the ordering region $q_{B\text{opt}} > 0$, the following ordering condition is present

$$\omega_1(x,t) < \frac{-V_x(\mu_S - \zeta - r)}{{\sigma_S}^2 V_{xx}}. \tag{39}$$

At the same time, if $(q_{B\text{opt}}, 0)$ is the optimal control of the static optimization problem, the value of the static problem of $(q_{B\text{opt}}, 0)$ must be better than that of (0, 0), which is

$$U(x,t) + G(q_{B\text{opt}}, 0) \frac{\partial V}{\partial x} + \frac{1}{2} H(q_{B\text{opt}}, 0) \frac{\partial^2 V}{\partial x^2} > U(x,t) + G(0,0) \frac{\partial V}{\partial x} + \frac{1}{2} H(0,0) \frac{\partial^2 V}{\partial x^2},$$

where

$$G(0,0) \equiv r\omega_0 + \omega_1(\mu_S - \zeta) + (P - S_1\bar{\omega})D,$$

and

$$H(0,0) = {\sigma_S}^2 \omega_1^2.$$

Expanding $G(q_{B\text{opt}}, 0)$, $H(q_{B\text{opt}}, 0)$, $G(0,0)$ and $H(0,0)$ and rearranging the items yields

$$-rFV_x + (\mu_S - \zeta - r) q_{B\text{opt}} V_x + \frac{{\sigma_S}^2}{2} \left(q_{B\text{opt}}^2 + 2\omega_1 q_{B\text{opt}}\right) V_{xx} > 0. \tag{40}$$

Substituting

$$q_{Bopt}(x,t) = \frac{-V_x(\mu_S - \zeta - r)}{{\sigma_S}^2 V_{xx}} - \omega_1(x,t),$$

into the inequality (40), we could further simplify it as

$$-rFV_x - \frac{{\sigma_S}^2}{2} V_{xx}\left(-\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}} - \omega_1(x,t)\right)^2 > 0. \tag{41}$$

Then from that inequality we can derive another ordering condition apparent as

$$\omega_1(x,t) < -\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{{\sigma_S}^2 V_{xx}}}. \tag{42}$$

So far the ordering conditions for the inventory can be summarized as

$$\omega_1(x(t+), t+) = q_{Bopt}(x,t) + \omega_1(x,t) = -\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}} > 0, \tag{43}$$

$$\omega_1(x,t) < -\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{{\sigma_S}^2 V_{xx}}}, \tag{44}$$

$$\frac{\partial^2 V}{\partial x^2} < 0. \tag{45}$$

The aforementioned ordering conditions yield the optimal inventory strategies for a risk-averse manager at time t when the total portfolio value is x. If the inventory value is below $-\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{{\sigma_S}^2 V_{xx}}}$, then placing an order to raise the inventory value to the level of $-\frac{(\mu_S - \zeta - r)V_x}{{\sigma_S}^2 V_{xx}}$ is optimal.

2.3.2. The optimal selling policy

Similarly, in the selling region $\Omega_S = \{(q_B = 0, q_S) \,|\, q_S > 0\}$, the static optimization problem is defined as

$$\sup_{q_S > 0}\left(U(x,t) + G(0,q_S)\frac{\partial V}{\partial x} + \frac{1}{2}H(0,q_S)\frac{\partial^2 V}{\partial x^2}\right), \tag{46}$$

where

$$G(0,q_S) \equiv r\{\omega_0 + (1-\eta)q_S\} + (\omega_1 - q_S)(\mu_S - \zeta) + (P - S_1\bar{\omega})D,$$

and

$$H(0,q_S) = {\sigma_S}^2(\omega_1 - q_S)^2.$$

The first-order necessary conditions of the static optimization problem with respect to q_S are given by

$$-(\mu_S - \zeta - r(1-\eta))\frac{\partial V}{\partial x} - {\sigma_S}^2(\omega_1 - q_S)\frac{\partial^2 V}{\partial x^2} = 0 \tag{47}$$

and

$$\frac{\partial^2 V}{\partial x^2} < 0. \tag{48}$$

Rearranging (47) gives

$$q_{Sopt}(x,t) = \frac{V_x(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2 V_{xx}} + \omega_1(x,t). \tag{49}$$

In the selling region $q_{Sopt} > 0$, we have the following selling condition

$$\omega_1(x,t) > -\frac{V_x(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2 V_{xx}}.$$

Similarly, we also have the condition that the static optimal value of the control $(0, q_{Sopt})$ must be better than that of the control $(0, 0)$, which is

$$U(x,t) + G(0,q_{Sopt})\frac{\partial V}{\partial x} + \frac{1}{2}H(0,q_{Sopt})\frac{\partial^2 V}{\partial x^2} > U(x,t) + G(0,0)\frac{\partial V}{\partial x} + \frac{1}{2}H(0,0)\frac{\partial^2 V}{\partial x^2}.$$

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By expanding $G\left(0, q_{\mathrm{Sopt}}\right)$, $H\left(0, q_{\mathrm{Sopt}}\right)$, $G(0,0)$ and $H(0,0)$, the inequality can be reduced as

$$-\frac{{\sigma_S}^2}{2} V_{xx} \left(\frac{(\mu_S - \zeta - (1-\eta)\, r)\, V_x}{{\sigma_S}^2 V_{xx}} + \omega_1(x,t) \right)^2 > 0.$$

The above inequality will always be correct in the selling region under the condition of $q_{\mathrm{Sopt}} > 0$. Therefore the selling point is characterized by

$$\omega_1(x(t+), t+) = \omega_1(x,t) - q_{\mathrm{Sopt}}(x,t) = -\frac{V_x(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2 V_{xx}} > 0, \tag{50}$$

$$\omega_1(x,t) > -\frac{V_x(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2 V_{xx}}, \tag{51}$$

$$\frac{\partial^2 V}{\partial x^2} < 0. \tag{52}$$

For a risk-averse manager, when the inventory value is larger than $-\frac{V_x(\mu_S-\zeta-r(1-\eta))}{{\sigma_S}^2 V_{xx}}$, the optimal strategy is to sell some inventory back to the supplier to reduce the inventory value to the level of $-\frac{V_x(\mu_S-\zeta-r(1-\eta))}{{\sigma_S}^2 V_{xx}}$.

2.3.3. The optimal inventory policy

The aforementioned ordering and selling conditions yield the optimal inventory strategies for a risk-averse manager at time t when the total portfolio value is x. If the inventory value is below $-\frac{(\mu_S-\zeta-r)V_x}{{\sigma_S}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{{\sigma_S}^2 V_{xx}}}$, then placing an order to raise the inventory value to the level of $-\frac{(\mu_S-\zeta-r)V_x}{{\sigma_S}^2 V_{xx}}$ is optimal. On the contrary, if the inventory value is larger than $-\frac{V_x(\mu_S-\zeta-r(1-\eta))}{{\sigma_S}^2 V_{xx}}$, the optimal strategy is to sell some inventory back to the supplier to reduce the inventory value to the level of $-\frac{V_x(\mu_S-\zeta-r(1-\eta))}{{\sigma_S}^2 V_{xx}}$.

3. Inventory model under a VaR constraint

In order to exercise proper risk control over the portfolio value, we will impose the VaR constraint on the inventory portfolio. Given a small time interval Δt, we define the loss of the portfolio during the interval $[t+, t+\Delta t]$ by

$$\Delta X(t) = X(t+\Delta t) - X(t+).$$

Given the level of significance z, the VaR definition is

$$\Pr(\Delta X(t) \le -\mathrm{VaR}_t) = z.$$

Therefore the definition of VaR implies

$$\mathrm{VaR}_t = -E_t[\Delta X(t)] - \Phi^{-1}(z)\sqrt{\mathrm{Cov}_t[\Delta X(t)]},$$

where $\Phi^{-1}(\cdot)$ is the standard normal distribution function.

In the case with one kind of raw material and one kind of product, the conditional mean and variance can be calculated as

$$\begin{aligned} E_t[\Delta X(t)] &= r\{\omega_0(t) + (1-\eta)\, q_S(t) - q_B(t) - F\Theta[q_B(t)]\}\,\Delta t \\ &\quad + (\omega_1(t) + q_B(t) - q_S(t))\,(\mu_S - \zeta)\,\Delta t + [P(t) - s_1\bar{\omega}]\, D(t)\Delta t, \end{aligned}$$

$$\mathrm{Cov}_t[\Delta X(t)] = {\sigma_S}^2\,(\omega_1(t) + q_B(t) - q_S(t))^2\,\Delta t.$$

If we restrict the value at risk below the level R, the VaR constraint is given by

$$a_1(\omega_1 + q_B - q_S) + a_2 q_S + a_3\Theta(q_B) + b \le R, \tag{53}$$

where

$$a_1 = -(\mu_S - \zeta - r)\,\Delta t - \Phi^{-1}(z)\sigma_S\sqrt{\Delta t}, \qquad a_2 = r\Delta t\eta, \qquad a_3 = r\Delta tF,$$
$$b = -[P - s_1\bar{\omega}]\, D\Delta t - rx\Delta t.$$

Thus the optimal inventory problem under value at risk constraint is given by

$$\sup_{q_B, q_S} E\left[\int_0^T U(\tau, X(\tau))\, d\tau + W(T, X(T))\right], \tag{54}$$

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subject to

$$dX(t) = r\{\omega_0(t) + (1-\eta)q_S(t) - q_B(t) - F\Theta[q_B(t)]\}dt + [\omega_1(t) + q_B(t) - q_S(t)] \times (\mu_S - \zeta)dt + [P(t) - S_1(t)\bar{\omega}]D(t)dt + [\omega_1(t) + q_B(t) - q_S(t)]\sigma_S dB(t), \quad (55)$$

$$a_1(\omega_1 + q_B - q_S) + a_2 q_S + a_3\Theta(q_B) + b \le R, \quad (56)$$

$$\omega_1(t+) = \omega_1(t) + q_B(t) - q_S(t) > 0. \quad (57)$$

3.1. The optimal ordering policy

For an ordering point as ($q_B > 0, q_S = 0$), the VaR constraint of the portfolio is calculated as

$$\mathrm{VaR}_t = a_1(\omega_1 + q_B) + a_3 + b,$$

where

$$a_1 = -(\mu_S - \zeta - r)\Delta t - \Phi^{-1}(z)\sigma_S\sqrt{\Delta t}, \qquad a_3 = r\Delta t F, \qquad b = -[P - s_1\bar{\omega}]D\Delta t - rx\Delta t.$$

Notice that

$$a_3 + b = -[P - s_1\bar{\omega}]D\Delta t - r(x - F)\Delta t.$$

For a manufacturer, $P(t)$ is the price of the product at time t and $S_1(t)\bar{\omega}$ is the cost of the raw material for the product. It is reasonable for us to assume that

$$P(t) \ge S_1(t)\bar{\omega}.$$

Obviously, the ordering cost \$$F$ must be significantly smaller than the total portfolio value x. Thus we have the following result

$$a_3 + b = -[P - s_1\bar{\omega}]D\Delta t - r(x - F)\Delta t \le 0.$$

Thus, we next only consider the case with $a_1 > 0$ because the VaR of the portfolio will always be negative and the constraint will not be active when $a_1 \le 0$. If the VaR is limited to below the level R, we have

$$\omega_1 + q_B \le (R - b - a_3)/a_1. \quad (58)$$

Then the ordering condition determined by the VaR constraint is

$$\omega_1 \le (R - b - a_3)/a_1.$$

Combining the ordering conditions without the VaR constraint, the optimal inventory ordering strategies under a VaR constraint can be summarized as

$$\omega_1(x(t+), t+) = q_{B\mathrm{opt}}(x,t) + \omega_1(x,t) = \min\left\{-\frac{(\mu_S - \zeta - r)V_x}{\sigma_S{}^2 V_{xx}}, \frac{R - b - a_3}{a_1}\right\}, \quad (59)$$

$$\omega_1(x,t) < \min\left\{-\frac{(\mu_S - \zeta - r)V_x}{\sigma_S{}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{\sigma_S{}^2 V_{xx}}}, \frac{R - b - a_3}{a_1}\right\}, \quad (60)$$

$$\frac{\partial^2 V}{\partial x^2} < 0. \quad (61)$$

3.2. The optimal selling policy

For a selling point as ($q_B = 0, q_S > 0$), the VaR constraint of the portfolio is calculated as

$$\mathrm{VaR}_t = a_1\omega_1 - (a_1 - a_2)q_S + b \le R,$$

where

$$a_1 = -(\mu_S - \zeta - r)\Delta t - \Phi^{-1}(z)\sigma_S\sqrt{\Delta t}, \qquad a_2 = r\Delta t\eta, \qquad b = -[P - s_1\bar{\omega}]D\Delta t - rx\Delta t.$$

In a situation where $a_1\omega_1 + b > R$, and $a_1 - a_2 \le 0$, the VaR cannot be controlled under the given level R because q_S must be positive. We can summarize the optimal inventory selling strategies under a VaR constraint as

$$\omega_1(x(t+), t+) = \omega_1(x,t) - q_{S\mathrm{opt}}(x,t) = \min\left\{-\frac{(\mu_S - \zeta - (1-\eta)r)V_x}{\sigma_S{}^2 V_{xx}}, \frac{R - b - a_2\omega}{a_1 - a_2}\right\},$$

$$\omega_1(x,t) > \min\left\{\frac{-V_x(\mu_S - \zeta - r(1-\eta))}{\sigma_S{}^2 V_{xx}}, \frac{R - b}{a_1}\right\},$$

$$\frac{\partial^2 V}{\partial x^2} < 0.$$

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3.3. The optimal inventory policy

We note

$$r_B = \min\left\{-\frac{(\mu_S - \zeta - r)\,V_x}{\sigma_S{}^2 V_{xx}} - \sqrt{\frac{-2rFV_x}{\sigma_S{}^2 V_{xx}}}, \frac{R - b - a_3}{a_1}\right\},$$
$$R_B = \min\left\{-\frac{(\mu_S - \zeta - r)\,V_x}{\sigma_S{}^2 V_{xx}}, \frac{R - b - a_3}{a_1}\right\},$$
$$r_S = \min\left\{\frac{-V_x\,(\mu_S - \zeta - r\,(1-\eta))}{\sigma_S{}^2 V_{xx}}, \frac{R - b}{a_1}\right\},$$
$$R_S = \min\left\{-\frac{(\mu_S - \zeta - r\,(1-\eta))\,V_x}{\sigma_S{}^2 V_{xx}}, \frac{R - b - a_2\omega_1}{a_1 - a_2}\right\}.$$

For a risk-averse manager, which implies $\frac{\partial^2 V}{\partial x^2} < 0$, the optimal inventory policy is determined by the variables (r_B, R_B, r_S, R_S). At time t when the total portfolio value is x, if the inventory value $\omega_1(x,t)$ is below r_B, then the manager places an order to raise the inventory value to the level of R_B. On the contrary, if the inventory value $\omega_1(x,t)$ is higher than r_S, then the manager sells the amount of inventory necessary to reduce the inventory value to the level of R_S.

3.4. Solution algorithm

The approach is illustrated for one raw material and one product, that is $n = 1, m = 1$, but can be extended for n inventories and m products. Thus the optimal inventory problem is given by

$$\sup_{q_B, q_S} E\left[\int_0^T U\,(t, X(t))\,dt\right],$$

subject to

$$\begin{aligned} dX(t) \;=\; & r\,\{\omega_0(t) + (1-\eta)\,q_S(t) - q_B(t) - F\Theta\,[q_B(t)]\}\,dt + [\omega_1(t) + q_B(t) - q_S(t)] \\ & \times [(\mu_S - \zeta)\,dt + \sigma_S dB(t)] + [P(t) - S_1(t)\,\bar{\omega}]\,D(t)dt, \end{aligned}$$
$$a_1\,(\omega_1 + q_B - q_S) + a_2 q_S + a_3\Theta\,(q_B) + b \le R,$$
$$\omega_1 + q_B - q_S > 0.$$

Define the optimal value function as

$$V\,(x,t) = \sup_{q_B, q_S} E\left[\int_t^T U\,(\tau, X\,(\tau))\,d\tau\right].$$

We have proven that the optimal problem is equivalent to finding a solution to the HJB-equation

$$\frac{\partial V}{\partial t}\,(x,t) + \sup_{q_B, q_S}\left(U\,(x,t) + G\,(q_B, q_S)\,\frac{\partial V}{\partial x} + \frac{1}{2}H\,(q_B, q_S)\,\frac{\partial^2 V}{\partial x^2}\right) = 0, \tag{62}$$

with the boundary condition

$$V(x,T) = 0, \qquad V(0,t) = 0,$$

where

$$G\,(q_B, q_S) \equiv r\,\{\omega_0 + (1-\eta)\,q_S - q_B - F\Theta\,(q_B)\} + (\omega_1 + q_B - q_S)\,(\mu_S - \zeta) + (P - s_1\bar{\omega})\,D,$$

and

$$H\,(q_B, q_S) = \sigma_S{}^2\,(\omega_1 + q_B - q_S)^2\,.$$

In the following computation, the utility function is defined to a power function of value

$$U(t,x) = e^{-\delta t}x^{\gamma}, \quad \delta > 0,\ 0 < \gamma < 1. \tag{63}$$

Under this type of utility function, the form of the value function is

$$V(x,t) = e^{-\delta t}h(t)x^{\gamma}, \quad \delta > 0,\ 0 < \gamma < 1.$$

Neglecting the derivatives of h with respect to x, we have

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t} h(x,t) x^{\gamma-1}, \qquad \frac{\partial^2 V}{\partial x^2} = \gamma(\gamma-1) e^{-\delta t} h(x,t) x^{\gamma-2}, \tag{64}$$

$$\frac{\partial V}{\partial t} = e^{-\delta t} h'(x,t) x^{\gamma} - \delta e^{-\delta t} h(x,t) x^{\gamma}. \tag{65}$$

Substituting the trial function into the HJB-equation reduces it to a Bernoulli equation for $h(t)$ which is an ordinary differential equation. Substituting the derivatives into Eq. (62), dividing by $e^{-\delta t}x^{\gamma}$ and rearranging give

$$h'(x,t) + A\left(q_{Bopt}, q_{Sopt}, x\right) h(x,t) + 1 = 0 \tag{66}$$

with the terminal condition

$$h(x,T) = 0,$$

where

$$A\left(q_{Bopt}, q_{Sopt}, x\right) = \gamma \frac{G\left(q_{Bopt}, q_{Sopt}\right)}{x} + \frac{1}{2}\frac{\gamma(\gamma-1) H\left(q_{Bopt}, q_{Sopt}\right)}{x^2} - \delta.$$

In the case without VaR constraint, substituting Eqs. (64) and (65) to (38) and (49), yields the reduced form for q_{Bopt} and q_{Sopt} as

$$q_{Bopt}(x, \omega_1, t) = \frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)} - \omega_1, \tag{67}$$

$$q_{Sopt}(x, \omega_1, t) = \omega_1 - \frac{x(\mu_S - \zeta - (1-\eta) r)}{{\sigma_S}^2(1-\gamma)}. \tag{68}$$

Furthermore, when there is no VaR constraint, in the ordering region $\Omega_B = \{(q_B, q_S = 0) \,|q_B > 0\}$, we have

$$q_{Bopt}(x,t) = \frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)} - \omega_1(x,t), \tag{69}$$

$$\omega_1(x,t) < \frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)} - \sqrt{\frac{2rFx}{(1-\gamma)\sigma^2}}. \tag{70}$$

In the selling region $\Omega_S = \{(q_B = 0, q_S) \,|q_S > 0\}$, we have

$$q_{Sopt}(x,t) = \omega_1(x,t) - \frac{(\mu_S - \zeta - (1-\eta) r) x}{{\sigma_S}^2(1-\gamma)}, \tag{71}$$

$$\omega_1(x,t) > \frac{(\mu_S - \zeta - (1-\eta) r) x}{{\sigma_S}^2(1-\gamma)}. \tag{72}$$

Finally, in the no truncation region $\Omega_{NT} = \{(q_B = 0, q_S = 0)\}$, we have

$$\frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)} - \sqrt{\frac{2rFx}{(1-\gamma)\sigma^2}} \le \omega_1(x,t) \le \frac{(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2(1-\gamma)}. \tag{73}$$

This unconstrained solution will be used as an initial guess to solve the problem. If we impose a VaR constraint to control the risk, the optimal inventory policy is denoted as (r_B, R_B, r_S, R_S), where

$$r_B = \min\left\{\frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)} - \sqrt{\frac{2rFx}{(1-\gamma){\sigma_S}^2}}, \frac{R - b - a_3}{a_1}\right\},$$

$$R_B = \min\left\{\frac{x(\mu_S - \zeta - r)}{{\sigma_S}^2(1-\gamma)}, \frac{R - b - a_3}{a_1}\right\},$$

$$r_S = \min\left\{\frac{(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2(1-\gamma)}, \frac{R - b}{a_1}\right\},$$

$$R_S = \min\left\{\frac{(\mu_S - \zeta - r(1-\eta))}{{\sigma_S}^2(1-\gamma)}, \frac{R - b - a_2\omega_1}{a_1 - a_2}\right\}.$$

Dividing the computational time horizon $[0, T]$ into a grid of N_t points and omitting (x, t) in all variables for the simplicity of notation, the algorithm can be summarized as follows:

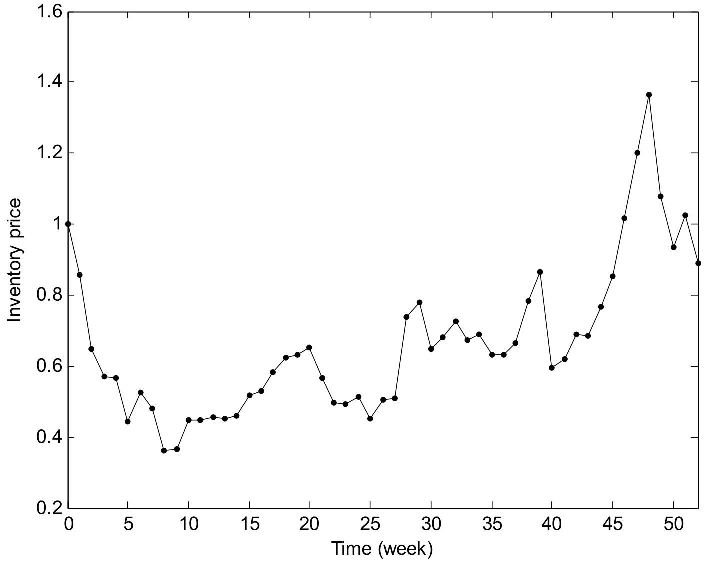

Fig. 1. Price sequence of the raw material.

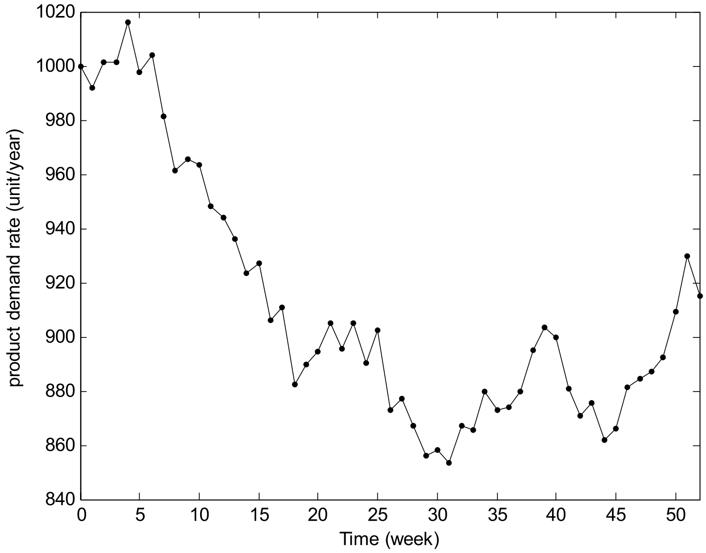

Fig. 2. Demand sequence of the product.

(1) At $t = 0$, we set $\omega_1(0) = 0, X(0) = x$.
(2) Calculate $q_{Bopt}(0), \omega_0\,(0+), \omega_1\,(0+)$ and $X\,(0+)$ by

$$q_{Bopt}(0) = \frac{x\,(\mu_S - r)}{{\sigma_S}^2\,(1-\gamma)} - \omega_1(0), \qquad \omega_1\,(0+) = \omega_1(0) + q_{Bopt}(0) = \frac{x\,(\mu_S - r)}{{\sigma_S}^2\,(1-\gamma)},$$

$$\omega_0\,(0+) = x - \omega_1(0) - q_{Bopt}(0) - F = x - \frac{x\,(\mu_S - r)}{{\sigma_S}^2\,(1-\gamma)} - F,$$

$$X\,(0+) = \omega_0\,(0+) + \omega_1\,(0+) = x - F.$$

(3) For $t_k = [\Delta t, \ldots, (N_t - 1)\Delta t, N_t \Delta t = T]$, generating two sequences of standard Brownian motion $B\,(t_k)$ and $\tilde{B}\,(t_k)$ randomly, the price process and the product demand process can be simulated by

$$S_1\,(t_k) = S_1(0)\exp\left\{\left(\mu_S - \frac{1}{2}{\sigma_S}^2\right)t + \sigma_S B\,(t_k)\right\},$$

$$D\,(t_k) = \mu_D + \sigma_D \tilde{B}\,(t_k)\,.$$

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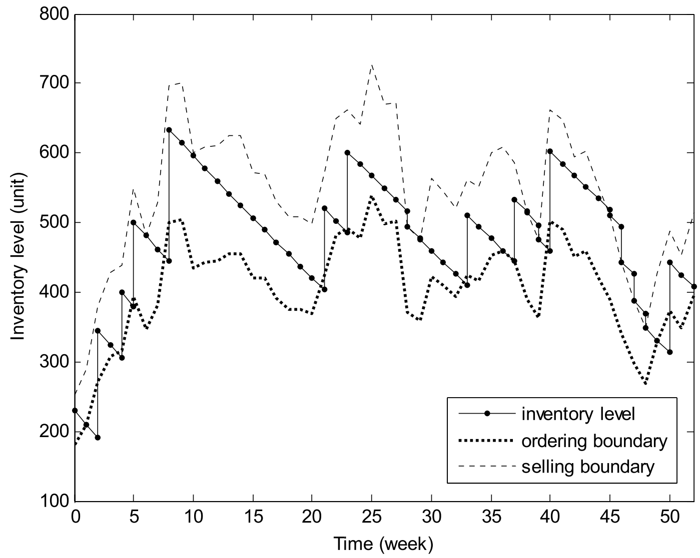

Fig. 3. Optimal inventory strategy without VaR constraint.

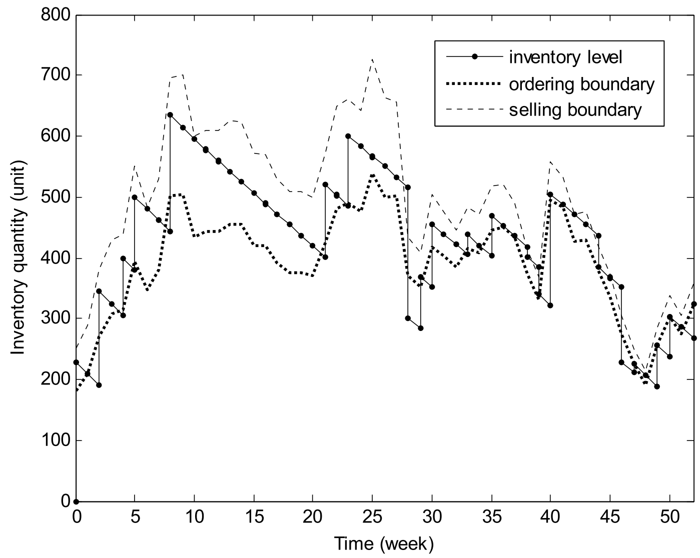

Fig. 4. Optimal inventory strategy under a VaR constraint.

(4) Then $\omega_0(t_k)$, $\omega_1(t_k)$ and $X(t_k)$ can be calculated by

$$\omega_0(t_k) = \omega_0(t_{k-1}+)\, e^{r\Delta t} + P(t_{k-1})\,\Delta t,$$

$$\omega_1(t_k) = \frac{\omega_1(t_{k-1}+)}{S_1(t_{k-1})} S_1(t_k) - D(t_{k-1})\,\bar{\omega} S_1(t_{k-1})\,\Delta t,$$

$$X(t_k) = \omega_0(t_k) + \omega_1(t_k).$$

(5) Calculate (r_B, R_B, r_S, R_S) by

$$r_B = \min\left\{\frac{X(t_k)(\mu_S - \zeta - r)}{\sigma_S{}^2(1-\gamma)} - \sqrt{\frac{2rFX(t_k)}{(1-\gamma)\,\sigma_S{}^2}}, \frac{R - b - a_3}{a_1}\right\},$$

$$R_B = \min\left\{\frac{X(t_k)(\mu_S - \zeta - r)}{\sigma_S{}^2(1-\gamma)}, \frac{R - b - a_3}{a_1}\right\},$$

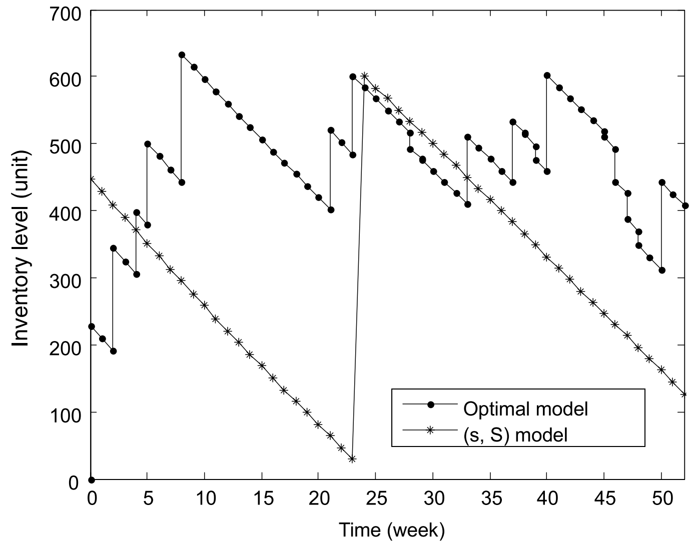

Fig. 5. Inventory level (unit).

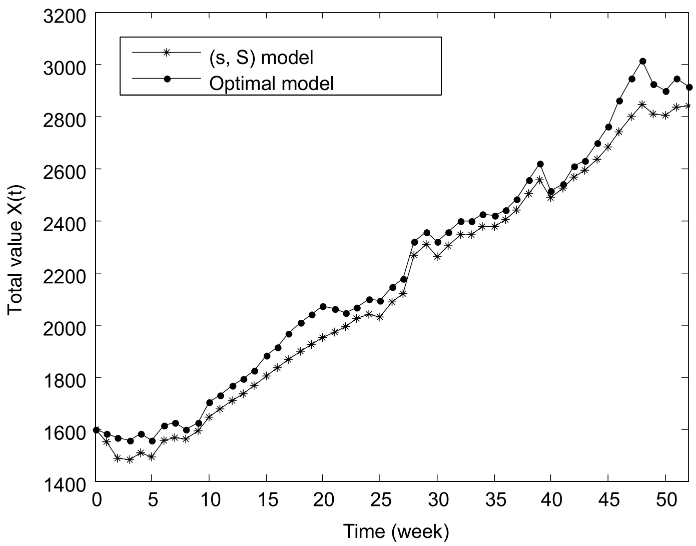

Fig. 6. Total portfolio value $X(t)$.

$$r_S = \min\left\{\frac{X(t_k)(\mu_S - \zeta - r(1-\eta))}{\sigma_S{}^2(1-\gamma)}, \frac{R-b}{a_1}\right\},$$
$$R_S = \min\left\{\frac{X(t_k)(\mu_S - \zeta - r(1-\eta))}{\sigma_S{}^2(1-\gamma)}, \frac{R-b-a_2\omega_1(t_k)}{a_1-a_2}\right\}.$$

If $\omega_1(t_k) < r_B(t_k)$, place an order for the raw material so as to

$$q_{Bopt}(t_k) = R_B - \omega_1(t_k), \qquad \omega_1(t_k+) = \omega_1(t_k) + q_{Bopt}(t_k) = R_B(t_k),$$
$$\omega_0(t_k+) = \omega_0(t_k) - q_{Bopt}(t_k) - F, \qquad X(t_k+) = X(t_k) - F.$$

If $\omega_1(t_k) > r_S(t_k)$, sell raw materials so as to

$$q_{Sopt}(t_k) = \omega_1(t_k) - R_S, \qquad \omega_1(t_k+) = \omega_1(t_k) - q_{Sopt}(t_k) = R_S(t_k),$$
$$\omega_0(t_k+) = \omega_0(t_k) + (1-\eta)\, q_{Sopt}(t_k), \qquad X(t_k+) = X(t_k) - \eta q_{Sopt}(t_k).$$

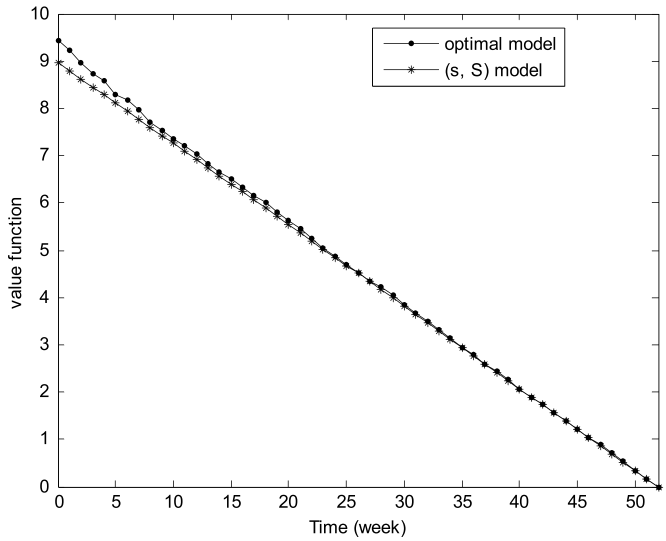

Fig. 7. Value function $V(x, t)$.

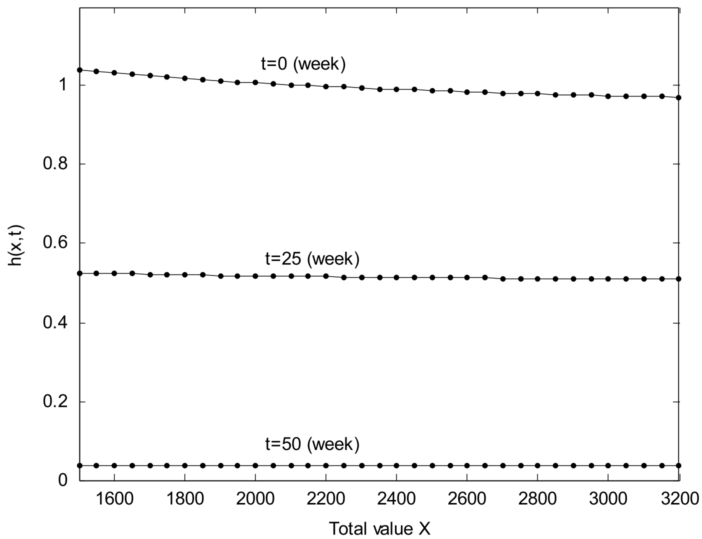

Fig. 8. $h(x, t)$ over x at different time.

If $r_B \leq \omega_1(t_k) \leq r_S$, we have

$$p_{\text{opt}}(t_k) = q_{\text{opt}}(t_k) = 0, \qquad \omega_1(t_k+) = \omega_1(t_k), \qquad \omega_0(t_k+) = \omega_0(t_k),$$
$$X(t_k+) = \omega_0(t_k+) + \omega_1(t_k) = X(t_k).$$

(6) Set $h(T) = 0$ and for $t_k = [(N_t - 1)\Delta t, \ldots, \Delta t, 0]$, we have

$$h(t_k) = h(t_{k+1}) + (A(t_k) * h(t_{k+1}) + 1) * \Delta t.$$

4. Illustrative examples

This section has four objectives. First, the optimal strategies for the inventory model with and without VaR constraint are presented. Second, the results of the (s, S) models are compared. Third, the behaviors of the optimal policies under VaR constraint are illustrated. Finally, the impacts of parameters including ordering, selling and holding costs are discussed.

A matlab program was written to implement the above procedure. The time horizon we considered is one year, $T = 1$. We assume the firm evaluates the VaR constraint once per week, which means that $\Delta t \approx 1/52$. The parameters in the utility

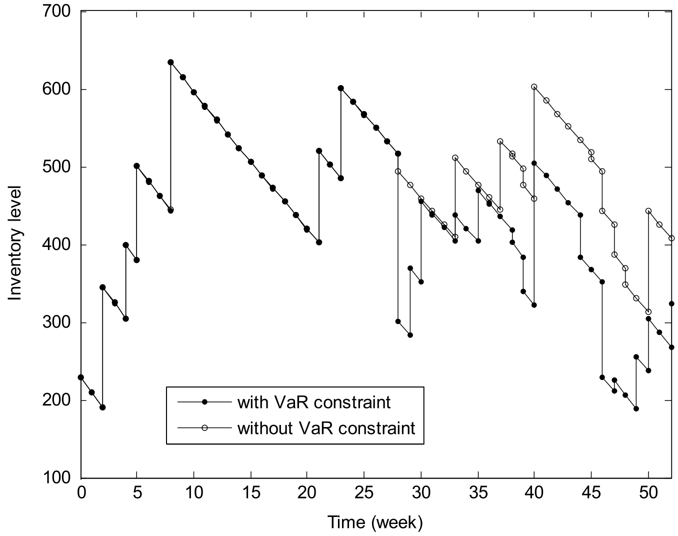

Fig. 9. Compare the inventory level with and without VaR constraint.

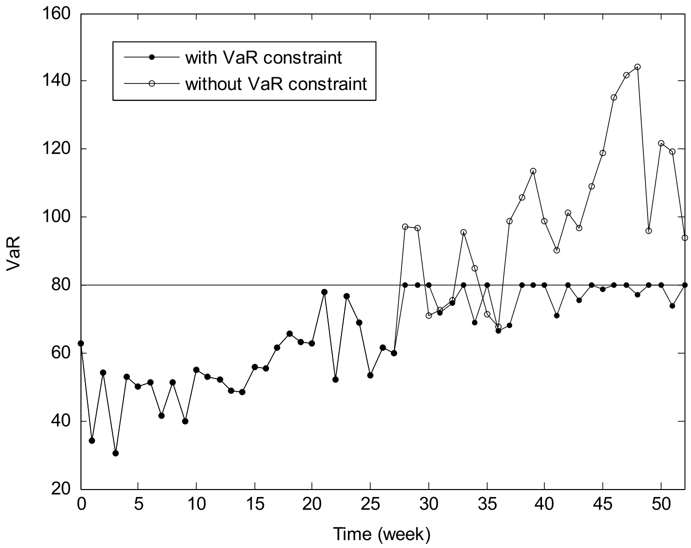

Fig. 10. VaR sequence at different time.

function are taken to be $\delta = 0.2$, $\gamma = 0.3$. For the VaR constraint, the maximum loss is limited to $R = 80$ with a probability of $k = 0.01$. In addition, the parameters of demand rate are assumed to be $\mu_D = 1000$, $\sigma_D = 100$ and the product price is $P(t) = 2$. The initial capital is $X(0) = 1600$ and the price of the material at $t = 0$ is $S_1(0) = 1$. Finally, the stochastic process of the price is chosen arbitrarily with $\sigma_S = 1$, $\mu_S = 0.25$ and the risk free rate is $r = 0.05$.

4.1. Optimal inventory strategies

The fixed order cost per non-zero order is $F = 10$ and the proportional selling cost per unit of inventory value is $\eta = 0.2$. Figs. 1 and 2 display the generated time series of the price of raw material $S_1(t)$ as well as the demand quantity $D_1(t)$ of the product. Figs. 3 and 4 illustrate the inventory levels, ordering and selling boundaries of the inventory model with and without VaR constraint. From the optimal inventory strategies illustrated in the figures it can be concluded that if the inventory level is below the ordering boundary, then the manager places an order to raise the inventory value to the optimal level; on the contrary, if the inventory level is higher than the ordering boundary, then the manager sells the amount of inventory necessary to reduce the inventory value to the optimal level.

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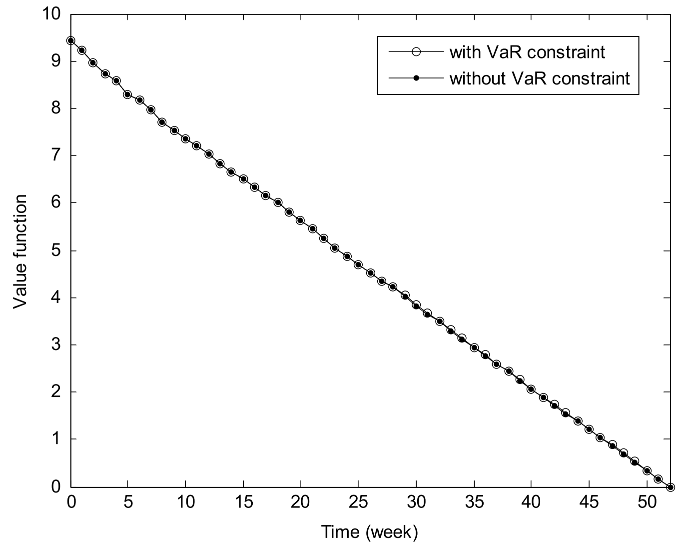

Fig. 11. Value function $V(x, t)$.

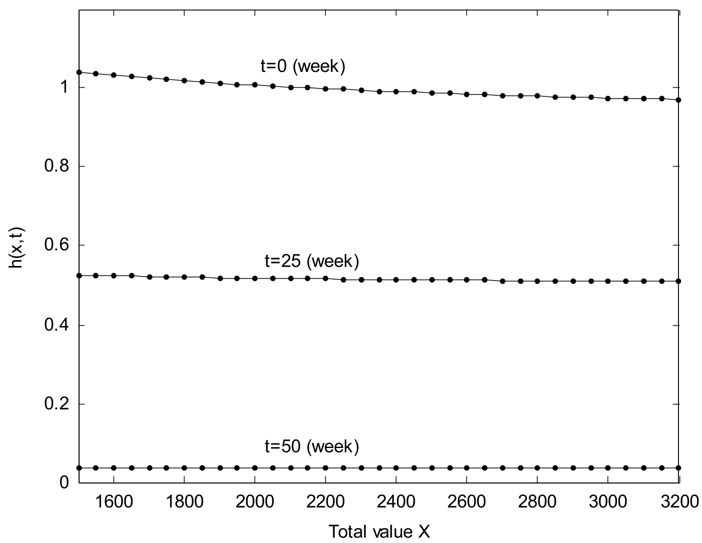

Fig. 12. $h(x, t)$ over x at different time.

4.2. Comparison with the (s, S) model

We compare the inventory level (Fig. 5), the total portfolio value (Fig. 6) and the value function (Fig. 7) between our optimal model and the classical (s, S) model. Since there is no lead time and the inventory review period is Δt, we calculate (s, S) as

$$s = D(t)\Delta t, \qquad S = EOQ = \sqrt{\frac{2FD(t)}{\zeta S_1(t)}}.$$

From Fig. 5, we find that the inventory level of the (s, S) model is independent on the price of the raw material (except at the ordering point), while in the present model the inventory level is positively related to the price sequence. That coincides with our common sense understanding. Figs. 6 and 7 prove that our optimal model is better than the (s, S) model from both the aspects of portfolio value and utility value. Fig. 8 depicts the $h(x, t)$ function over x at different times. It illustrates that $h(x, t)$ changes little for different x values, so our neglect of the derivatives of $h(x, t)$ with respect to x is reasonable.

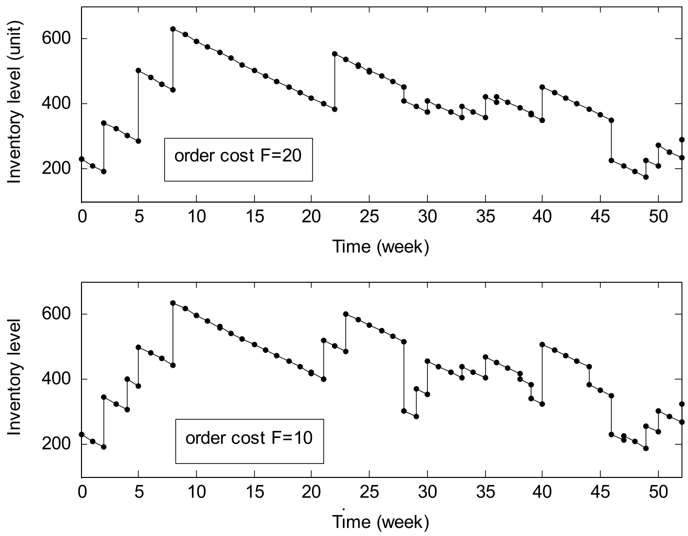

Fig. 13. Inventory level of different order cost.

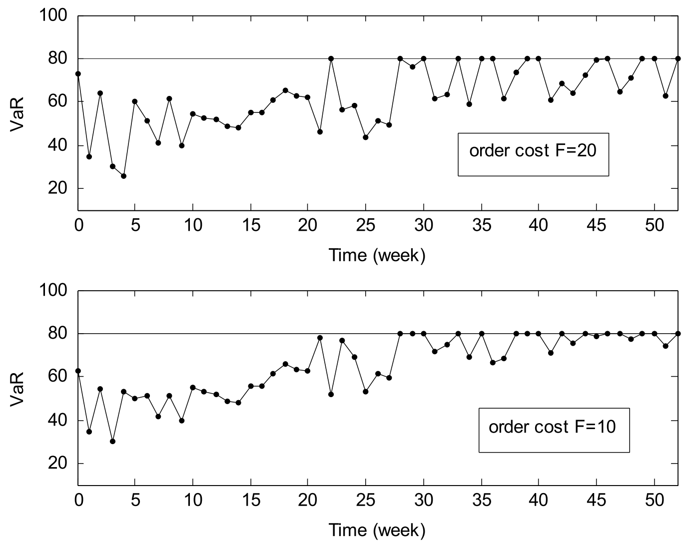

Fig. 14. VaR for different order cost.

4.3. Effects of the VaR constraint on the optimal inventory policy

This will be compared to the inventory level (Fig. 9) and VaR sequence (Fig. 10) both with and without VaR constraint in the optimal inventory model. The figures show that good control over the investment in the raw material inventory quantity is achieved and the inventory level is reduced in order to fulfill the VaR constraint. When the VaR constraint is inactive the portfolio allocation follows the unconstrained solution; as the portfolio value increases, the VaR constraint becomes active (Fig. 10) and allocates less to the raw material inventory. The points where the constraints become active produce kinks in the curve, but the risk is well controlled through the VaR constraint. Figs. 11 and 12 depict the value function $V(x, t)$ and the $h(x, t)$ function over time. Fig. 11 shows that the difference between the value functions in the cases with and without a VaR constraint is very small. Thus we can conclude that imposing a VaR constraint will not significantly affect the optimal value function.

4.4. Effects of cost parameters on the optimal inventory policy

This section shows how ordering and holding costs affect inventory policies. As conjectured, the ordering frequency is reduced when the ordering cost is increased, which is confirmed in Fig. 13. With augmentation of the ordering cost, the loss

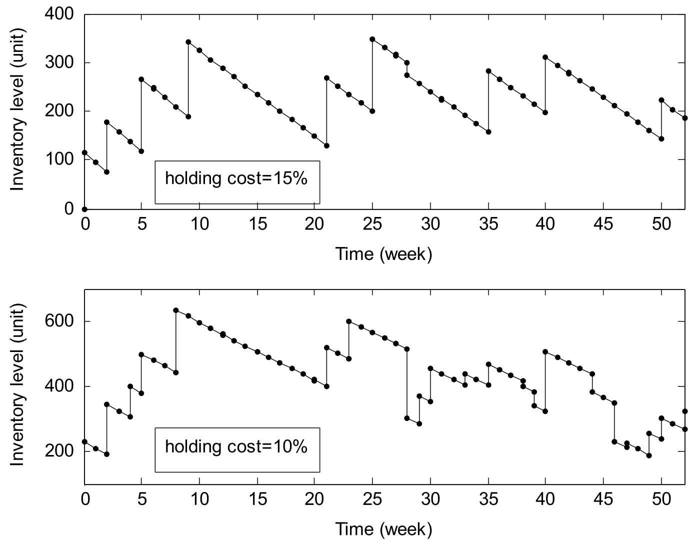

Fig. 15. Inventory level for different holding cost.

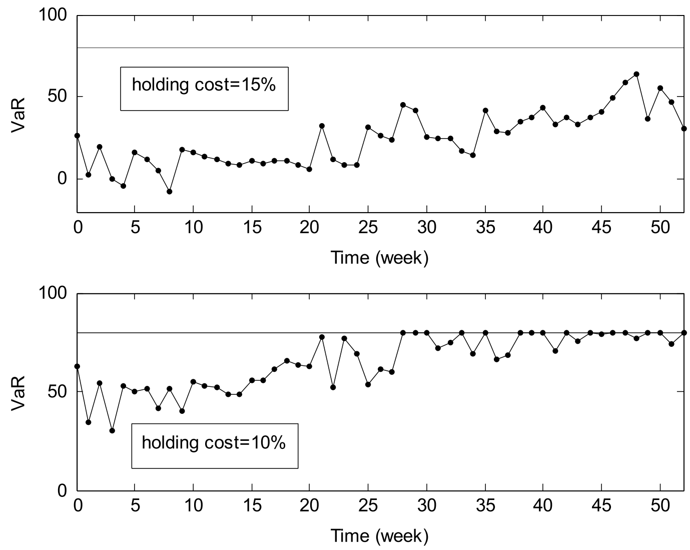

Fig. 16. VaR for different holding cost.

risk of the total portfolio value is also enlarged, so the VaR constraint becomes active at earlier points in time (Fig. 4). We conclude from Fig. 5 that when the holding cost increases from 5% to 15%, the inventory level is greatly reduced (see Figs. 14 and 15). The low inventory level directly results in a low VaR level, which is represented in Fig. 16.

5. Conclusion

This paper presented optimal inventory policies under a VaR constraint. By considering the raw material inventories as a kind of risky investment, we optimized the portfolio consisting of the risk free bank account and the raw material inventory. The problem is formulated as a continuous time stochastic control problem with fixed ordering costs and proportional selling and holding costs. We have proven that the optimal problem is reduced to solve the Hamilton–Jacobi–Bellman equation by applying the dynamic programming technique and the stochastic control theory. The VaR constraint is imposed continuously to control the risk of the portfolio. The optimal inventory strategy under a VaR constraint is summarized as a continually reviewed policy (r_B, R_B, r_S, R_S). In the end, we proposed a numerical algorithm to solve the constrained optimal stochastic problem with a power-law utility function. From the numerical results, we find that risk is effectively reduced where holdings in raw material inventories are optimally decreased.

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