A Model of Type Theory in Simplicial Sets

A brief introduction to Voevodsky's Homotopy Type Theory

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Abstract

As observed in [HS] identity types in intensional type theory endow every type with the structure of a *weak higher dimensional groupoid*. The simplest and oldest notion of weak higher dimensional groupoid is given by *Kan complexes* within the topos **sSet** of simplicial sets. This was observed around 2006 independently by V. Voevodsky and the author.

The aim of this note is to describe in an accessible way how simplicial sets organize into a model of Martin-Löf type theory. Moreover, we explain Voevodsky's *Univalence Axiom* which holds in this model and implements the idea that isomorphic types are identical as suggested in [HS]. A full exposition of the theory will be given in a longer article by Voevodsky which is still in preparation, but see [VV]. The current note just gives a first introduction to this circle of ideas.

1 Simplicial Sets and Kan complexes

Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and monotone maps between them. We write **sSet** for the topos $\mathbf{Set}^{\mathbf{\Delta}^{\mathsf{op}}}$ of *simplicial sets*. We write $[n]$ for the ordinal $n{+}1$ and $\Delta[n]$ for the corresponding representable object in **sSet**. For $0 \leq k \leq n$ we write $i^n_k : \Lambda_k[n] \hookrightarrow \Delta[n]$ for the inclusion of the k-th *horn* $\Lambda_k[n]$ into $\Delta[n]$ which is obtained by removing the interior and the face opposite to vertex k. As described e.g. in [GJ] there is an obvious faithful functor $|\cdot|$ from $\mathbf{\Delta}$ into the category **Sp** of spaces and continuous maps. This induces the *singular* functor $\mathcal{S} : \mathbf{Sp} \to \mathbf{sSet}$ sending X to $\mathbf{Sp}(|{-}|, X)$ which has a left adjoint $\mathcal{R}$ called *geometric realization*. The objects in the image of $\mathcal{R}$ are the so-called CW complexes which can be obtained by glueing simplices in a way as described by some simplicial set. The objects in the image of $\mathcal{S}$ are the so called *Kan complexes* which can be characterized in a more combinatorial way as we will describe in the next paragraph.

On **sSet** there is a well known *Quillen model structure* whose class $\mathcal{C}$ of *cofibrations* consists of all monos, whose class $\mathcal{W}$ of *weak equivalences* consists of all maps $f : X \to Y$ whose geometric realization $|f| : |X| \to |Y|$ is a homotopy equivalence[1] and whose class $\mathcal{F}$ of fibrations consists of all *Kan fibrations*, i.e. maps $a : A \to I$ in **sSet** with $i^n_k \perp a$ for all $n, k \in \mathbb{N}$.[2] It is shown in [GJ] that a simplicial set X is a *Kan complex* iff $X \to 1$ is a Kan fibration. Moreover, a map $f : X \to Y$ between Kan complexes is a weak equivalence iff f is a homotopy equivalence, i.e. there is a map $g : Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.[3]

In **sSet** one can develop a fair amount of homotopy theory and as shown in [GJ] inverting weak equivalences in **sSet** gives rise to the same *homotopy category* as inverting weak equivalences in **Sp**. Thus, from a homotopy point of view **sSet** and **Sp** are different ways of speaking about the same thing. However, the "combinatorial" topos **sSet** is in many respects much nicer then the "geometric" category **Sp**. This we exploit when interpreting intensional Martin-Löf type theory in **sSet**.

2 Homotopy Model for Type Theory

Since **sSet** is a topos and thus locally cartesian closed it provides a model of extensional type theory (since **sSet** contains also a natural numbers object N). In order to obtain a non-trivial interpretation of identity types we restrict families of types to be *Kan fibrations*. Apparently $\mathcal{F}$ contains all isos and is closed under composition and pullbacks along arbitrary morphisms in **sSet**. Using the fact that trivial cofibrations are stable under pullbacks along Kan fibrations one easily establishes that

Theorem 2.1 *Kan fibrations are closed under* Π*, i.e. whenever* $a : A \to I$ *and* $b : B \to A$ *are in* $\mathcal{F}$ *then* $\Pi_a(b)$ *is in* $\mathcal{F}$*, too.*

[1] i.e. there exists a continuous map $g : |Y| \to |X|$ such that both composita are homotopy equivalent to the identities $\mathrm{id}_{|X|}$ and $\mathrm{id}_{|Y|}$, respectively

[2] $f \perp g$ means that for every commuting square $kf = gh$ there is a diagonal filler, i.e. a map d with $df = h$ and $gd = k$ as in

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] \arrow[ur, "d"] & \cdot \end{tikzcd}$$

[3] For $f, g : A \to B$ we write $f \sim g$ iff there is a map $h : \Delta[1] \times A \to B$ with $h(0, -) = f$ and $h(1, -) = g$.

For interpreting equality on X we factor the diagonal $\delta_X : X \to X \times X$ as

$$
\begin{tikzcd}
X \arrow[r, "r_X"] \arrow[dr, "\delta_X"'] & \mathrm{Id}(X) \arrow[d, "p_X"] \\
& X \times X
\end{tikzcd}
$$

where $p_X \in \mathcal{F}$ and $r_X \in \mathcal{C} \cap \mathcal{W}$ which is possible since $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Quillen model structure. The Kan fibration p_X will serve as interpretation of $x, y : X \vdash \mathrm{Id}_X(x, y)$ as suggested in [AW]. For families of types as given by a Kan fibration $a : A \to I$ one factors the fibrewise diagonal $\delta_a : A \to A \times_I A$ in an analogous way. However, there is a problem since such factorisations are in general not stable under pullbacks. To overcome this problem we introduce universes à la Martin-Löf.

As described in [VV] a universe in **sSet** is a Kan fibration $p_U : \widetilde{U} \to U$. We write $\mathcal{D}_U$ for the class of Kan fibrations which can be obtained as pullbacks of p_U along some map in **sSet**. In [VV] Voevodsky has shown how such a universe induces a contextual category $CC[p_U]$ which interprets dependent sums if $\mathcal{D}_U$ is closed under composition and which interprets dependent products if $\mathcal{D}_U$ is closed under Π.

Some time ago M. Hofmann and the author observed how to lift a Grothendieck universe $\mathcal{U}$ in **Set** to a type theoretic universe $p_U : \widetilde{U} \to U$ in a presheaf topos $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$. The object U is defined as

$$U(I) = \mathcal{U}^{(\mathcal{C}/I)^{\mathsf{op}}} \qquad U(\alpha) = \mathcal{U}^{\Sigma_\alpha^{\mathsf{op}}}$$

where for $\alpha : J \to I$ the functor $\Sigma_\alpha : \mathcal{C}/J \to \mathcal{C}/I$ is postcomposition with α. The idea behind this definition is that $\mathcal{U}^{(\mathcal{C}/I)^{\mathsf{op}}}$ is equivalent to the full subcategory of $\widehat{\mathcal{C}}/\mathsf{Y}(I)$ on those maps whose fibres are small in the sense of $\mathcal{U}$. The presheaf $\widetilde{U}$ is defined as

$$\widetilde{U}(I) = \{\langle A, a\rangle \mid A \in U(I) \text{ and } a \in A(\mathrm{id}_I)\}$$

and

$$\widetilde{U}(\alpha)(\langle A, a\rangle) = \langle U(\alpha)(A), A(\alpha \stackrel{\alpha}{\to} \mathrm{id}_I)(a)\rangle$$

for $\alpha : J \to I$ in $\mathcal{C}$. The map $p_U : \widetilde{U} \to U$ sends $\langle A, a\rangle$ to A. One easily checks that p_U is *generic* for maps with fibres small in the sense of $\mathcal{U}$, i.e. these maps are up to isomorphism precisely those which can be obtained as pullback of p_U along some map in $\widehat{\mathcal{C}}$.

Now for $\mathcal{C} = \Delta$ we adapt this idea in such a way that p_U is generic for Kan fibrations with fibres small in the sense of $\mathcal{U}$. For this purpose we redefine U as

$$U([n]) = \{A \in \mathcal{U}^{(\mathbf{\Delta}/[n])^{\mathsf{op}}} \mid P_A \text{ is a Kan fibration}\}$$

where $P_A : \mathsf{Elts}(A) \to \Delta[n]$ is obtained from A by the Grothendieck construction. For maps α in $\mathbf{\Delta}$ we can define $U(\alpha)$ as above since Kan fibrations are stable under pullbacks. We also define $\widetilde{U}$ and p_U using the same formulas as above but understood as restricted to U in its present form.

Theorem 2.2 *The simplicial set U is a Kan complex.*

This has been shown in [VV] for a different construction of the universe. A simpler proof of Theorem 2.2 for the above construction of U has been found recently by A. Joyal.

Theorem 2.3 *The map $p_U : \widetilde{U} \to U$ is universal for Kan fibrations which are small in the sense of $\mathcal{U}$.*

Proof: For showing that p_U is a Kan fibration suppose

$$\begin{tikzcd} \Lambda_k[n] \arrow[r, "a"] \arrow[d, hook, "i_k^n"'] & \widetilde{U} \arrow[d, "p_U"] \\ \Delta[n] \arrow[r, "A"'] & U \end{tikzcd}$$

commutes. Since the pullback of p_U along A is the Kan fibration $P_A : \mathsf{Elts}(A) \to \Delta[n]$ there exists a diagonal filler $\overline{a} : \Delta[n] \to \widetilde{U}$ making

$$\begin{tikzcd} \Lambda_k[n] \arrow[r, "a"] \arrow[d, hook, "i_k^n"'] & \widetilde{U} \arrow[d, "p_U"] \\ \Delta[n] \arrow[r, "A"'] \arrow[ur, "\overline{a}"'] & U \end{tikzcd}$$

commute.

For showing that p_U is universal suppose that $a : A \to I$ is a Kan fibration small in the sense of $\mathcal{U}$. Then one gets a as pullback of p_U along the morphism $A : I \to U$ sending $x \in I([n])$ to an $\mathcal{U}$-valued presheaf over $\mathbf{\Delta}/[n]$ which via the Grothendieck construction is isomorphic to x^*a. $\square$

Thus p_U provides us with a universe in **sSet** which is closed under dependent sums and products. Since $N = \Delta(\mathbb{N})$ is a small Kan complex this universe also hosts the natural numbers object N.

For interpreting identity types in this universe we consider the map $\delta_{\widetilde{U}} : \widetilde{U} \to \widetilde{U} \times_U \widetilde{U}$ with $\pi_i \circ \delta_{\widetilde{U}} = \mathrm{id}$ for $i = 0, 1$ where

$$\begin{tikzcd} \widetilde{U} \times_U \widetilde{U} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{U} \arrow[d, "p_U"] \\ \widetilde{U} \arrow[r, "p_U"'] & U \end{tikzcd}$$

and consider a factorisation

$$\begin{tikzcd} \widetilde{U} \arrow[r, "r_{\widetilde{U}}"] \arrow[dr, "\delta_{\widetilde{U}}"'] & \mathrm{Id}_{\widetilde{U}} \arrow[d, "p_{\widetilde{U}}"] \\ & \widetilde{U} \times_U \widetilde{U} \end{tikzcd}$$

with $p_{\widetilde{U}} \in \mathcal{F}$ and $r_{\mathcal{U}} \in \mathcal{C} \cap \mathcal{W}$.

For interpreting the eliminator J for Id-types we pull back the whole situation along the projection p from the generic context

$$\Gamma \equiv C : \mathrm{Id}_{\widetilde{U}} \to_U U^* U, d : \Pi_U(r^*_{\widetilde{U}} C)$$

to U. Since p is a Kan fibration and pullbacks along Kan fibrations preserve weak equivalences we have $p^* r_{\widetilde{U}} \in \mathcal{C} \cap \mathcal{W}$. Let $q : \widetilde{C} \to p^*\mathrm{Id}_{\widetilde{U}}$ be the interpretation of the type family $\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x,y,z) \vdash C(x,y,z)$ and $d : p^*\widetilde{U} \to \widetilde{C}$ be the interpretation of the term $\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x,y,z) \vdash d(x) : C(x,y,z)$. Obviously, we have $q \circ d = p^* r_{\widetilde{U}}$. Since q is a Kan fibration and $p^* r_{\widetilde{U}} \in \mathcal{C} \cap \mathcal{W}$ by the defining properties of Quillen model structures there is a map J making the diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "q"] & p^*\mathrm{Id}_{\widetilde{U}} \arrow[d, equal] \\ p^*\widetilde{U} \arrow[u, "d"] \arrow[r, "p^* r_{\widetilde{U}}"'] & p^*\mathrm{Id}_{\widetilde{U}} \arrow[ul, "J"'] \end{tikzcd}$$

commute. This map J serves as interpretation of the eliminator for identity types associated with types in the universe U.

NB Factoring $\delta_{\widetilde{U}}$ and d relative to the generic context Γ prevents one from proving that r and J satisfy BCC, i.e. are stable under pullbacks. However, since trivial cofibrations are not stable under arbitrary pullpacks the instantiations of $r_{\widetilde{U}}$ are not guaranteed to be trivial cofibrations. This problem, however, can be avoided when choosing $r_{\widetilde{U}}$ as the canonical map $\widetilde{U} \to \widetilde{U}^{\Delta[1]}$ in the fibre over U because such maps are stable under arbitary pullbacks.

If one starts from the universe $\mathcal{U} = \{\emptyset, \{\emptyset\}\}$ one obtains a universe $p_U : \widetilde{U} \to U$ where $U([n])$ is the set of those subobobjects $m : P \hookrightarrow \Delta[n]$ which are Kan fibrations. One easily shows by induction over n that such subobjects are trivial in the sense that m is an isomorphism whenever P is not initial.[4] Thus p_U is obtained by restricting $\top : 1 \rightarrowtail \Omega_{\mathbf{sSet}}$ along the mono $2 \rightarrowtail \Omega_{\mathbf{sSet}}$. When interpreting Prop by this p_U one obtains a boolean, 2-valued proof-irrelevant interpreation of Coq.

[4]It is an open question, however, whether for any Kan fibration $p : E \to B$ its image is a union of connected components of B.

Finally we want to emphasize that the model sketched in this section implements the idea that types are *weak higher dimensional groupoids* which are here realized as Kan complexes. Moreover, it keeps the interpretation of Prop from the naive model in **Set**.

3 Voevodsky's Univalence Axiom

We now give the formulation of Voevodsky's Univalence Axiom which as shown in [VV] holds in the model described in the previous section. For this purpose we first introduce a few abbreviations

$$
\begin{aligned}
&\mathrm{iscontr}(X{:}\mathrm{Set}) = (\Sigma x{:}X)(\Pi y{:}Y)\ \mathrm{Id}_X(x,y)\\
&\mathrm{hfiber}(X,Y{:}\mathrm{Set})(f{:}X{\to}Y)(y{:}Y) = (\Sigma x{:}X)\ \mathrm{Id}_Y(f(x),y)\\
&\mathrm{isweq}(X,Y{:}\mathrm{Set})(f{:}X{\to}Y) = (\Pi y{:}Y)\ \mathrm{iscontr}(\mathrm{hfiber}(X,Y,f,y))\\
&\mathrm{Weq}(X,Y{:}\mathrm{Set}) = (\Sigma f{:}X{\to}Y)\ \mathrm{isweq}(X,Y,f)
\end{aligned}
$$

Using the eliminator J for identity types one constructs a canonical map

$$\mathrm{eqweq}(X,Y{:}\mathrm{Set}) : \mathrm{Id}_{\mathrm{Set}}(X,Y) \to \mathrm{Weq}(X,Y)$$

The Univalence Axiom[5] then claims that all maps $\mathrm{eqweq}(X,Y)$ are themselves weak equivalences, i.e.

$$\mathrm{UnivAx} : (\Pi X,Y{:}\mathrm{Set})\ \mathrm{isweq}(\mathrm{eqweq}(X,Y))$$

which, alas, doesn't seem to have any computational meaning.

Notice, moreover, that $\mathrm{isweq}(X,Y)(f)$ is equivalent to

$$\mathrm{isiso}(X,Y)(f) \equiv (\Sigma g{:}Y{\to}X)\big((\Pi x{:}X)\mathrm{Id}_X(g(fx),x)\big) \times \big((\Pi y{:}Y)\mathrm{Id}_Y(f(gy),y)\big)$$

which formally says that f is an isomorphism but due to the interpretation of identity types in **sSet** rather claims that f is a *homotopy equivalence*. This equivalence is provable in type theory without the Univalence Axiom (see [VV] for a Coq file containing a machine checked proof). It is in accordance with the fact that in **sSet** morphisms to Kan complexes are weak equivalences iff they are homotopy equivalences. The type theoretic argument may be seen as an example for a "synthetic" version of a theorem in homotopy theory.

A suprising consequence of the Univalence Axiom is that it allows one to prove the function extensionality principle

$$\big((\Pi x{:}X)\,\mathrm{Id}_Y(fx,gx)\big) \to \mathrm{Id}_{X\to Y}(f,g)$$

for $f,g : X \to Y$ (see [VV] for a Coq file containing a machine checked proof).

[5] The name "univalent" insinuates that the universe Set contains up to propositional equality only one representative of each class of weakly equivalent types.

4 Conclusion

The model of intensional type theory in **sSet** realizes the idea from [HS] that propositional equality of types should coincide with isomorphism. It is not clear so far what are the benefits of this identification for the formalisation of category theory.

However, as described in various Coq files from [VV] one may use type theory as an internal language for developing homotopy theory synthetically. The basic idea is that the type of paths from x to y in X is given by $\mathrm{Id}_X(x, y)$. For information on more recent developments within this rapidly developing new field of research consult the blog `http://homotopytypetheory.org`.

References

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