A Method to Symbolically Compute Convolution Integrals

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis presents a method for computing symbolic solutions of a certain class of improper integrals related to convolutions of Mellin transforms. Important integrals that fall into this category are integral transforms such as the Fourier, Laplace, and Hankel transforms. The method originated in a presentation by Salvy, However, many of the details of the method were absent. We present the method of Salvy in full which computes a linear homogeneous differentail equation which is satisfied by the integral in question. A theory of contour integrals is introduced that is related to the contour definition of Meijer G functions. This theory is used to prove the correctness of the method of Salvy and also gives a way to compute regions of validity for the solutions computed. We then extend the method to compute symbolic solutions of the integral along with where the solutions are valid.

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Contents

Chapter 1

Introduction

An integral transform $\mathcal{T}[f(x);t]$ is an integral defined by

$$\mathcal{T}[f(x);t] = \int_{t_1}^{t_2} K(t,x)f(x)dx$$

for some function $K(t,x)$ (called the Kernel function). There are many different integral transforms, each determined by choosing $K(t,x)$ and the limits of integration t_1, and t_2. The integral transform acts on an input function $f(x)$ and outputs another function $\mathcal{T}[f(x);t]$. Some kernel functions have an associated inverse kernel function which define an inverse transform. Under certain conditions, when an inverse transform is applied to the transform of a function the result is the original function.

Integral transforms are used in many areas of mathematical science. Their usefulness comes from the ability to map functions of one domain to functions of a different domain. For example, the Fourier transform maps functions from a time domain to a frequency domain. Often, a problem can be quite difficult to solve in its orginal domain but if the problem is transformed to a different domain then the problem becomes much easier to solve. The solution in the new domain can then be transformed back into the original domain using the inverse of the transform. An example of this is using the Laplace transform to convert the problem of solving a differential equation into the problem of solving an algebraic equation.

In this thesis, we will present a method for the computation of integrals of the form:

$$\int_0^\infty f(t)g(xt)dt. \tag{1.1}$$

Integrals such as (1.1) are of particular interest as many important integral transforms can be written in this form. Special cases of (1.1) are the Fourier transform, Laplace transform, and the lesser known Hankel transform. A more complete list, including definitions of the transforms, is given in Appendix A.

1.1 Main Results

In 2000, Salvy [16] presented an overview of a new algorithm that computes integrals in the form of (1.1). A very brief outline of the algorithm was given while most details were omitted. In this thesis, we present a complete description of the algorithm described by Salvy and fill in the missing details. This includes descriptions of the algorithms, required assumptions and associated proofs of correctness.

Currently, there exist methods in computer algebra systems for explicitly calculating integrals similar to the form of (1.1). However, when these methods fail how can we resolve the problem? Salvy presented the answer as computing an implicit solution. That is, we compute a differential equation such that the integral (1.1) is a solution of the differential equation.

For example, suppose one wishes to compute (1.1) where $f(x)$ and $g(x)$ both solve

$$xy''(x) + y'(x) + xy(x) = 0.$$

Salvy's algorithm can compute a differential equation satisfied by (1.1), which in this case would be

$$(x^3 - x)y'(x) + (x^2 - 1)y(x) = 0. \tag{1.2}$$

Unfortunately, for many integrals, knowing only the differential equations that are satisfied by $f(x)$ and $g(x)$ is not enough information to determine an equation like (1.2). We will often also require the behaviour of $f(x)$ and $g(x)$ at zero and infinity.

We present two methods for determining the differential equation satisfied by (1.1). Chapter 5 presents the first method which uses the theory of Mellin transforms. The method computes differential equations satisfied by integrals with the form of (1.1) under certain assumptions along the real x axis. Chapter 6 introduces a second method which is an extension of the first. The method relies on a theory that, in some sense, generalizes Mellin transforms. The extended method removes some of the assumptions required by the first method and, as well, computes equations which are valid in regions of the complex plane.

When computing these differential equations, care must be taken to identify the regions where they are valid. The integral being considered may only converge in certain regions of the complex plane. When using the first method, the theory of Mellin transforms guarantees that the differential equation computed is valid and the integral converges on the positive real line. However, it is not necessarily the case that one particular solution exists for the differential equation that is equal everywhere to the integral. For example, consider computing

$$I(x) = \int_0^\infty \mathrm{Ci}(t)\sin(xt)^2 dt.$$

The complete solution on the positive real line is

$$I(x) = \begin{cases} \frac{\pi}{8x} & \text{if } x > \frac{1}{2} \\ \frac{\pi}{8} & \text{if } x = \frac{1}{2} \\ 0 & \text{if } 0 < x < \frac{1}{2} \end{cases} . \tag{1.3}$$

Computing the differential equations satisfied by $\mathrm{Ci}(x)$ and $\sin(x)$ and running Salvy's algorithm with them as input gives the differential equation

$$(x^3 - x)y'(x) + (x^2 - 1)y(x) = 0.$$

While $I(x)$ does not equal any particular solution, it does satisfy the differential equation everywhere except at the point $x = \frac{1}{2}$. The integral is said to be a weak solution to the differential equation.

The second method computes differential equations valid in some region of the complex plane. The points of convergence for the integral must also be determined for this method as there is no guarantee that the integral will converge within the regions where the differential equation is valid. This is an artifact of using analytic continuation in the theory. The converse is also possible and the integral may converge outside the regions where the differential equation is valid. It is often possible to compute other differential equations that are valid in different regions. This makes it possible to still give a complete implicit solution in these cases.

Often, when computing an integral, one is given $f(x)$ and $g(x)$ as explicit closed form functions. Usually it is the case that when closed form input is given then a closed form output is desired. In this thesis we focus on computing closed form solutions of $I(x)$ given $f(x)$ and $g(x)$ in closed form. From the differential equations obtained by either of the above methods we will show how to obtain explicit closed form solutions of the integral. The solutions will possibly be weak solutions such as in (1.3). We also identify the regions where these solutions are valid.

1.2 Other Approaches

There are several approaches to computing integrals with the form (1.1) in modern computer algebra systems such as Maple or Mathematica. The first technique is to use the fundamental theorem of calculus. One would compute the antiderivative of the integrand, and use the fundamental theorem of calculus to compute the result from the antiderivative. The problem with this technique is that the fundamental theorem of calculus cannot be applied blindly. Many additional calculations have to be made, such as computing zeros of different functions. Symbolically, determining the zeros of many functions is often extremely difficult or impossible. There can also be an infinite number of zeros in the case of an oscillating function.

Another obvious technique to compute these integrals is to use look-up tables. A large number of integrals are precomputed and assembled into a table. To compute an integral, the integrand is looked up in the table, if it is found the solution is returned. This procedure can be improved in several ways. One way is to arrange the integrals into a standard form before the integrands are looked up. See [12] for an example of how to vastly improve look-up tables of integrals that are the definition of special functions. However, the method suffers from many drawbacks. For example, the tables are never complete and errors may exist in the table.

One nontrivial technique used by modern computer algebra systems is to use the Meijer G function formula for the integration of two Meijer G functions. The technique is described in Chapter 3. The method still relies on look-up tables, but instead of searching for the integrand itself, the two functions in the integrand are searched one at a time. Each search determines a way to express the function as a Meijer G function. Once both functions in the integrand are converted to Meijer G functions the integral formulas can be applied. The result of the formula is another Meijer G function which equals the integral [4, page 346]. Usually, users want results in terms of simpler functions, so computer algebra systems simplify the Meijer G functions using algorithms such as those presented in [15]. The Meijer G technique allows for the computation of a large number of the integrals that appear in integral tables such as [2], [3], [4] and [5].

The most modern technique in computer algebra systems for computing integrals with the form of (1.1) is known as differentiation under the integral. The technique also solves other types of integrals as well. Suppose L is a differential operator that annihilates $f(t)g(xt)$. From L another differential operator L' can be computed that annihilates $f(t)g(xt)$ and does not contain t. L' is made up of two parts, the first part $D_t \cdot P$ that has a common factor D_t, and the second part Q that does not contain D_t. Then

$$L' \cdot f(t)g(xt) = (D_t \cdot P + Q)\, f(t)g(xt) = 0$$

so,

$$\begin{aligned} 0 &= \int_0^\infty (D_t \cdot P + Q) \cdot f(t)g(xt)dt \\ &= Q \cdot \int_0^\infty f(t)g(xt)dt + [P \cdot f(t)g(xt)]_{t=0}^{\infty} . \end{aligned}$$

If $[P \cdot f(t)g(xt)]_{t=0}^{\infty} = 0$ then Q is a differential operator that annihilates the integral. This method has the inherent problem that it requires elimination in a non-commutative algebra, an expensive operation. Problems also exist for determining the regions where the solutions of the differential equations equal the integral. This technique originated in [19] and [20], also see [7],[8], [9], [11] and [10].

1.3 Outline

In Chapter 2 we present some mathematical foundations of holonomic functions and Mellin transforms. These are the fundamental background theories needed in later chapters. The Mellin transform plays a key role in the method we will present in Chapter 5. Chapter 3 gives an overview of the Meijer G technique for explicitly calculating integrals that are in the form of (1.1). The Meijer G technique gives the motivation for the ideas that are needed to extend the method presented in Chapter 5. In Chapter 4 we present some further mathematical background on asymptotic expansions of functions, formal solutions of differential equations, analytic continuation of Mellin transforms, how the analytic continuation of Mellin transforms of functions relates to the function's own asymptotic expansions and how to compute the asymptotic expansions of integrals with the form of (1.1). In Chapter 5, we present a method to compute integrals with the form (1.1) along the real line. Chapter 6 presents some new ideas to go beyond Mellin transforms and extends the method of Chapter 5. The method is extended to compute more integrals with the form of (1.1) and at the same time obtain solutions that are valid in some part of the complex plain instead of just along the real line. Finally, in Chapter 7 we summarize the results presented in the previous chapters and give some concluding remarks.

Chapter 2

Mathematical Foundations

2.1 The Mellin Transform

The Mellin transform is an integral transform named after Robert Hjalmar Mellin (1854-1933). It plays an important role in the relation between the theory of Gamma functions and special functions as well as the theory of asymptotic expansions and definite integration. Many of this section's results can be found in any book on integral transforms that includes a discussion on the Mellin transform. The results of this section primarily follow from [18] which proves the theorems below from first principles. We assume the reader is familiar with holomorphic, integrable ($\mathcal{L}(a,b)$), and locally integrable ($\mathcal{L}_{loc}(a,b)$) functions, as well as bounded variation. The integrals in this section are to be taken as Lebesgue integrals.

2.1.1 Definition of the Mellin Transform

Definition 2.1.1. The Mellin transform of a function, $f(x)$, is a locally integrable function given by

$$\mathcal{M}[f(x);s] = \int_0^\infty x^{s-1} f(x)dx,$$

when the integral exists. □

From the definition it is clear that the Mellin transform of a function will not always exist. Indeed the Mellin transform does not exist even for simple functions such as polynomials. At best, the Mellin transform will exist for some subset of the complex plane. The region where the Mellin transform converges absolutely is called the fundamental strip.

Theorem 2.1.1. *Let $f(x)$ be a function locally integrable on $(0,\infty)$ and let*

$$\begin{aligned} \alpha &= \inf\{a|f(x) = O(x^{-a}),\ \textit{as } x \to 0^+\} \textit{ and} \\ \beta &= \sup\{b|f(x) = O(x^{-b}),\ \textit{as } x \to \infty\}. \end{aligned}$$

Then $\mathcal{M}[f(x);s]$ will converge absolutely and be holomorphic everywhere in the region $\alpha < \Re(s) < \beta$. □

By Theorem 2.1.1, the Mellin transform of a function $f(x)$ will converge absolutely and be holomorphic in some vertical strip of the complex plane $\alpha < \Re(s) < \beta$. We denote this region $\langle \alpha, \beta \rangle$ and refer to this as the fundamental strip of $f(x)$.

Example 2.1.1. The Gamma function is given by,

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

The integral above is the Mellin transform of e^{-x}. The fundamental strip of e^{-x} can be obtained by observing its asymptotic expansions at zero and infinity. At zero, e^{-x} has a power series expansion,

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}. \tag{2.1}$$

At infinity e^{-x} does not have a power series expansion but it decreases faster than x^{-n} for integer n. Therefore,

$$\begin{aligned} \alpha = \inf\{a | e^{-x} = O(x^{-a}), \text{ as } x \to 0^+\} &= 0 \text{ and} \\ \beta = \sup\{b | e^{-x} = O(x^{-b}), \text{ as } x \to \infty\} &= \infty. \end{aligned}$$

By Theorem 2.1.1, e^{-x} has a fundamental strip $\langle \alpha, \beta \rangle = \langle 0, \infty \rangle$ and the Gamma function, $\Gamma(s)$ exists for any $\Re(s) > 0$. It is also possible to continue the Gamma function to the entire complex plain as a meromorphic function with poles at negative integers. Such extensions can be performed for a large class of Mellin transforms. □

2.1.2 Inversion of the Mellin Transform

The Mellin transform has the inversion formula defined by

$$f(x) = \frac{1}{2\pi i} \lim_{t \to \infty} \int_{c-it}^{c+it} x^{-s} \mathcal{M}[f(x);s] ds,$$

where c lies in the fundamental strip of $f(x)$. This is justified in the following two theorems.

Theorem 2.1.2. *Let $f(x)$, $x > 0$ be a function such that $y^{k-1} f(y)$ belongs to $\mathcal{L}(0, \infty)$, and let it be of bounded variation in the neighbourhood of the point $y = x$.*

If

$$\phi(s) = \int_0^\infty f(x)x^{s-1}dx,$$

then

$$\frac{1}{2}\left(f(x+0)+f(x-0)\right) = \frac{1}{2\pi i}\lim_{t\to\infty}\int_{k-it}^{k+it}\phi(s)x^{-s}ds.$$

□

Theorem 2.1.3. *Let $\phi(s)$ be a function such that $\phi(k+iu)$ belongs to $\mathcal{L}(-\infty,\infty)$, and let it be of bounded variation in the neighbourhood of the point $u=t$. If*

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\phi(s)x^{-s}ds,$$

then

$$\frac{1}{2}\left(\phi(k+i(x+0))+\phi(k+i(x-0))\right) = \lim_{\lambda\to\infty}\int_{\frac{1}{\lambda}}^{\lambda} f(x)x^{k+it-1}dx.$$

□

Proofs of Theorems 2.1.2 and 2.1.3 can be found in [18, page 46].

Example 2.1.2. The Gamma function is defined as

$$\Gamma(s) = \mathcal{M}[e^{-x};s].$$

By Theorem 2.1.2, we obtain an integral representation for e^{-x},

$$e^{-x} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}x^{-s}\Gamma(s)ds.$$

Theorem 2.1.2 requires that $y^{k-1}e^{-y}$ belongs to $\mathcal{L}(0,\infty)$. In Example 2.1.1, the fundamental strip for e^{-x} is given as $\langle 0,\infty\rangle$. By Theorem 2.1.1, $y^{k-1}e^{-y}$ belongs to $L(0,\infty)$ for every $k\in\langle 0,\infty\rangle$. □

Note that, in the above example, the integral representation for e^{-x} is only justified for $x>0$ by Theorem 2.1.2. However, e^{-x} is a very well-behaved function as it is an entire function. The question then becomes whether the integral

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}x^{-s}\Gamma(s)ds$$

in Example 2.1.2 converges to e^{-x} for any other points away from the positive real line. This will be addressed in the following two theorems.

Theorem 2.1.4. *Let $f(x)$ be a holomorphic function for $-\alpha < \arg(x) < \beta$, where $0 < \alpha \leq \pi$ and $0 < \beta \leq \pi$. Let $f(x) = O(|x|^{-a-\epsilon})$ as $|x| \to 0$ and $f(x) = O(|x|^{-b+\epsilon})$ as $|x| \to \infty$, where $a < b$, for any $-\alpha < \arg(x) < \beta$ and any $\epsilon > 0$. Then*

$$\phi(s) = \int_0^\infty f(x)x^{s-1}dx,$$

is a holomorphic function of s for $a < \Re(s) < b$, $\phi(k+it) = O(e^{-(\beta-\epsilon)t})$ as $t \to \infty$ and $\phi(k+it) = O(e^{(\alpha-\epsilon)t})$ as $t \to -\infty$ for every $\epsilon > 0$, uniformly for any $a < k < b$. Also

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \phi(s)x^{-s}ds$$

holds for $a < k < b$. □

Theorem 2.1.5. *Let $\phi(s)$ be a holomorphic function in the strip $a < \Re(s) < b$ where $a < b$, $\phi(k+it) = O(e^{-(\beta-\epsilon)t})$ as $t \to \infty$ and $\phi(k+it) = O(e^{(\alpha-\epsilon)t})$ as $t \to -\infty$ for every $\epsilon > 0$, uniformly for any $a < k < b$. Then*

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \phi(s)x^{-s}ds$$

is a holomorphic function of $x = re^{i\theta}$ for $-\alpha < \arg(x) < \beta$, $a < k < b$ and $f(x) = O(|x|^{-a-\epsilon})$ as $|x| \to 0$ and $f(x) = O(|x|^{-b+\epsilon})$ as $|x| \to \infty$, uniformly for any $-\alpha < \arg(x) < \beta$ and any $\epsilon > 0$. Also

$$\phi(s) = \int_0^\infty f(x)x^{s-1}dx$$

holds. □

Example 2.1.3. Writing $|e^{-x}|$ with $x = re^{i\theta}$ as

$$\begin{aligned} |e^{-re^{i\theta}}| &= |e^{-r(\cos(\theta)+i\sin(\theta))}| \\ &= |e^{-ri\sin(\theta)}||e^{-r\cos(\theta)}| \\ &= e^{-r\cos(\theta)} \end{aligned}$$

we observe as $|x| \to \infty$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then the function decreases in a uniform manner as $\cos(\theta) > 0$. However, if θ is not in the region $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the function will not decrease exponentially. Therefore, the order estimates of e^{-x} obtained in Example 2.1.1 are only valid in the region $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Note that the expansion of e^{-x} at zero in Example 2.1.1 actually converges to e^{-x} for all complex numbers. Thus, by Theorem 2.1.4, the Gamma function has the following property.

$$\Gamma(k+it) = \begin{cases} O(e^{-(\frac{\pi}{2}-\epsilon)t}) & \text{as } t \to \infty \\ O(e^{-(\frac{\pi}{2}\epsilon)t}) & \text{as } t \to -\infty \end{cases}$$

when $k > 0$. By Theorem 2.1.5, the integral

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} x^{-s}\Gamma(s)ds$$

converges to e^{-x} when $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$ and $k > 0$. □

2.1.3 Important Properties

The Mellin transform satisfies a number of translational and differential properties. All these results follow from the definition of the Mellin transform.

Theorem 2.1.6. *The Mellin transform satisfies the following properties:*

1. $\mathcal{M}[f(ax);s] = a^{-s}\mathcal{M}[f(x);s](a > 0)$
2. $\mathcal{M}[x^a f(x);s] = \mathcal{M}[f(x);s+a]$
3. $\mathcal{M}[f(x^a);s] = a^{-1}\mathcal{M}[f(x);s/a](a > 0)$
4. $\mathcal{M}[f(x^{-a});s] = a^{-1}\mathcal{M}[f(x);-s/a](a > 0)$
5. $\mathcal{M}[(\log(x))^n f(x);s] = \frac{d^n}{ds^n}\mathcal{M}[f(x);s]$

□

Similarly, there is a theorem for computing the Mellin transform of a function's derivative.

Theorem 2.1.7. *Let $f(x)$ be a function with fundamental strip $\langle \alpha, \beta \rangle$. Then,*

$$\mathcal{M}[f^{(n)}(x);s] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)}\mathcal{M}[f(x);s-n]$$

for the strip $\langle \alpha + n, \beta + n \rangle$. □

Parseval's formula is a central property used throughout this thesis. The formula is given in the following theorem.

Theorem 2.1.8. *Let $x^{1-k}f(x)$ and $x^k g(x)$ belong to $\mathcal{L}(0,\infty)$ and let*

$$I(x) = \int_0^\infty f(t)g(xt)dt.$$

Then $x^k I(x)$ belongs to $\mathcal{L}(0,\infty)$ and

$$I(x) = \int_{k-i\infty}^{k+i\infty} \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s} ds.$$

□

Corollary 2.1.1. *Let $f(x), g(x) \in \mathcal{L}_{loc}(0,\infty)$. Suppose the fundamental strip of $f(x)$ is $\langle \alpha, \beta \rangle$ and the fundamental strip of $g(x)$ is $\langle \gamma, \delta \rangle$. If the strips $\langle 1-\alpha, 1-\beta \rangle$ and $\langle \gamma, \delta \rangle$ overlap then*

$$\mathcal{M}[I(x); s] = \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$$

and the fundamental strip of $I(x)$ is the overlapping strip. Also

$$I(x) = \int_{k-i\infty}^{k+i\infty} \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s} ds$$

for any k in the overlapping strip. □

Example 2.1.4. Consider computing the Fourier sine transform

$$I(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(xt) dt.$$

The fundamental strip for $\sin(x)$ is $\langle -1, 0 \rangle$. Therefore, if the fundamental strip of $f(x)$ is $\langle \alpha, \beta \rangle$ and $(1-\beta, 1-\alpha) \cap (-1, 0) \neq \emptyset$ then by Corollary 2.1.1,

$$\begin{aligned} \mathcal{M}[I(x); s] &= \sqrt{\frac{2}{\pi}} \mathcal{M}[f(x); 1-s]\mathcal{M}[\sin(x); s] \\ &= \sqrt{\frac{2}{\pi}} \mathcal{M}[f(x); 1-s] \left(2^{-1+s}\sqrt{\pi} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(-\frac{1}{2}s+1)} \right) \end{aligned}$$

and $I(x)$ has a fundamental strip of $\langle \max\{1-\beta, -1\}, \min\{1-\alpha, 0\} \rangle$. Thus, by Theorem 2.1.2 or 2.1.4,

$$I(x) = \frac{1}{2\pi i} \sqrt{\frac{2}{\pi}} \lim_{t\to\infty} \int_{k-it}^{k+it} x^{-s} \mathcal{M}[f(x); 1-s] \left(2^{-1+s}\sqrt{\pi} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(-\frac{1}{2}s+1)} \right) ds,$$

for $k \in (\max\{1-\beta, -1\}, \min\{1-\alpha, 0\})$. Later in Chapter 4 we will demonstrate a method to compute $I(x)$ in this form using residue calculus. □

2.2 Holonomic Functions And Sequences

Holonomic functions and sequences play a large role in the ensuing chapters. In fact, in the subsequent chapters we will assume that all functions appearing in integrals are holonomic. The results found within this section are surveyed in [17]. This is where the description of the Maple package *gfun* can also be found. *gfun* is a Maple software module that contains all the procedures to manipulate holonomic functions described below. The theory of the package *Mgfun*, an extension to the *gfun* package, is described in [7], [8], [9], [11] and [10].

Definition 2.2.1. A function of a single variable is said to be holonomic if it satisfies a homogeneous linear differential equation with polynomial coefficients. □

Similarly a sequence can also be holonomic.

Definition 2.2.2. A sequence is said to be holonomic if it satisfies a linear recurrence with polynomial coefficients. □

It can be shown that if a function is holonomic, then its sequence of Taylor coefficients is a holonomic sequence. Conversely, if a sequence is holonomic, then its generating function is holonomic. While the previous property of holonomic functions and sequences is not relevant in later chapters it is important that holonomic functions and sequences do satisfy a large number of closure properties illustrated in the theorem below.

Theorem 2.2.1. *Suppose $f(x)$ and $g(x)$ are two functions then:*

1. *if $f(x)$ is an algebraic function, then $f(x)$ is holonomic,*
2. *if $f(x)$ and $g(x)$ are holonomic then, $f(x) + g(x)$ is holonomic,*
3. *if $f(x)$ and $g(x)$ are holonomic, then $f(x)g(x)$ is holonomic,*
4. *if $f(x)$ and $g(x)$ are holonomic, then the Hadamard product of $f(x)$ and $g(x)$ is holonomic,*
5. *if $f(x)$ is holonomic and $g(x)$ is algebraic, then $f(g(x))$ is holonomic,*
6. *if $f(x)$ is holonomic, then $f'(x)$ is holonomic, and*
7. *if $f(x)$ is holonomic, then $\int_c^x f(t)dt$ is holonomic.*

□

All the proofs of the properties of Theorem 2.2.1 (surveyed in [17]) are constructive and thus give rise to algorithms for computing all the listed operations on holonomic functions. A similar theorem exists for holonomic sequences [17].

Example 2.2.1. Consider proving $\sin^2(x)+\cos^2(x) = 1$. Note that $\sin(x)$ satisfies the differential equation

$$y''(x) + y(x) = 0$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$ and $\cos(x)$ satisfies the same differential equation with initial conditions $y(0) = 1$ and $y'(0) = 0$. Using the algorithm to compute the differential equation of a product of holonomic functions (property 3 of Theorem 2.2.1), we observe that $\sin^2(x)$ satisfies

$$y'''(x) + 4y'(x) = 0$$

with initial conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$. Similarly, $\cos^2(x)$ satisfies the same differential equation with initial conditions $y(0) = 1$, $y'(0) = 0$ and $y''(0) = -2$. Then, we use the algorithm to compute the differential equation of a sum of holonomic functions to obtain that $\sin^2(x) + \cos^2(x)$ satisfies

$$y'''(x) + 4y'(x) = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 0$. Since 1 is the solution to the previous differential equation and satisfies the initial conditions we show that $\sin^2(x) + \cos^2(x) = 1$. □

Chapter 3

The Meijer G Function

The Meijer G function was first defined by Cornelis Simon Meijer (1904 - 1974) in 1936. Its creation was an attempt to create a general function that included many other special functions as a special case. Other functions such as the generalized hypergeometric function and the MacRobert's E-Function were also created for the same reason. The Meijer G function is the most general of these functions as it includes them as special cases. Most special functions can be expressed in terms of the Meijer G function and the Gamma function. The definition of the Meijer G function and many of its properties are found in [4] and [13].

3.1 Definition

The Meijer G function is defined as

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}x^{-s}ds,$$

a contour integral with path C which will be defined below. The Meijer G function exists if:

1. $0 \le m \le q$, $0 \le n \le p$ with $m,n,p,q \in \mathbb{Z}_{\ge 0}$,
2. $a_j - b_k \notin \mathbb{Z}_{>0}$ for $j = 1,\ldots,n$ and $k = 1,\ldots,m$, and
3. $x \ne 0$.

The contour C is one of three possible paths:

1. A path from $-i\infty$ to $i\infty$ such that all poles of $\Gamma(b_j + s)$, $j = 1\ldots,m$ are on the left and all poles of $\Gamma(1 - a_k - s)$, $k = 1\ldots,n$ are on the right.

2. A loop starting and ending at ∞ encircling all poles of $\Gamma(1-a_k-s)$, $k = 1\dots,n$ exactly once in the negative direction and not encircling any poles of $\Gamma(b_j+s)$, $j=1\dots,m$.

3. A loop starting and ending at $-\infty$ encircling all poles of $\Gamma(b_j+s)$, $j = 1\dots,m$ exactly once in the positive direction and not encircling any poles of $\Gamma(1-a_k-s)$, $k=1\dots,n$.

The choice of path C is determined by which paths cause the integral to converge. To determine this, we consider the integrand of the integral

$$g(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}$$

and its behaviour at $s = -i\infty, i\infty, -\infty, \infty$. If the integrand is decreasing fast enough at the end points of C then the integral will exist because $g(s)$ is analytic on the rest of C.

To compute the behaviour of $g(s)$, we recall Stirling's formula:

$$\ln(\Gamma(s)) \sim \ln(s)s - s - \frac{1}{2}\ln(s)$$

as $s\to\infty$. To determine when the integral exists, if C is the path from $-i\infty$ to $i\infty$, consider the following inequality,

$$\left|\int_C g(s)x^{-s}ds\right| \le |x|^{\sigma}\int_C |g(s)|\, e^{\arg(x)|s|}ds$$

where σ is the $\Re(s)$ as $s\to i\infty, -i\infty$. The integral will exist if

$$\int_C |g(s)|\, e^{\arg(x)|s|}ds$$

exists. This integral will exist for $|\arg(x)| < \psi$ if ψ is the exponential decay of $g(s)$ as $s\to i\infty, -i\infty$.

From Stirling's formula, it is possible to obtain the estimates

$$\ln(\Gamma(a+si)) \sim -\frac{1}{2}y\pi - \frac{1}{2}\ln(y) + \ln(y)\Re(a)$$

as $s\to\infty$ and

$$\ln(\Gamma(a-si)) \sim -\frac{1}{2}y\pi - \frac{1}{2}\ln(y) - \ln(y)\Re(a)$$

$s\to\infty$. Therefore, every Gamma function on the top of the fraction in $g(s)$ contributes $\frac{1}{2}\pi$ to the exponential decay of $g(s)$ as $s\to i\infty, -i\infty$ and every Gamma function on the bottom contributes $-\frac{1}{2}\pi$. Thus, the integral exists for $|\arg(x)| < \frac{\pi}{2}(n+m-(p-n+q-m)) = \pi\left(n+m-\frac{p+q}{2}\right)$.

To determine if the integral exists for $|\arg(x)| = \pi\left(n+m-\frac{p+q}{2}\right)$ consider the algebraic decay of $g(s)$ as $s \to i\infty, -i\infty$. Using the previous two estimates, we have that $\prod_{j=1}^{m}\Gamma(b_j+s)$ contributes

$$\left(-\frac{1}{2}+\sigma\right)m + \sum_{j=1}^{m}\Re(b_j)$$

to the algebraic decay, $\prod_{j=1}^{n}\Gamma(1-a_j-s)$ contributes

$$\left(-\frac{1}{2}-\sigma\right)n + \sum_{j=1}^{n}(1-\Re(a_j)),$$

$\frac{1}{\prod_{j=n+1}^{p}\Gamma(a_j+s)}$ contributes

$$\left(\frac{1}{2}-\sigma\right)(p-n) - \sum_{j=n+1}^{p}\Re(a_j),$$

and $\frac{1}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}$ contributes

$$\left(\frac{1}{2}+\sigma\right)(q-m) - \sum_{j=m+1}^{q} 1-\Re(b_j).$$

Summing the above terms gives

$$(q-p)\left(\sigma-\frac{1}{2}\right) + \sum_{j=1}^{q}\Re(b_j) - \sum_{j=1}^{p}\Re(a_j). \tag{3.1}$$

Therefore, the integral exists for $|\arg(x)| = \pi\left(n+m-\frac{p+q}{2}\right)$ if (3.1)< -1.

The integral exists when C is the loop from ∞ back to ∞ if $G(s)$ decreases fast enough at ∞. From Stirling's formula, it is possible to obtain another two estimates

$$\ln\left(\Gamma(a+s)\right) \sim s\ln(s) - s - \frac{1}{2}\ln(s) + \Re(a)\ln(s)$$

as $s \to \infty$ and

$$\ln\left(\Gamma(a-s)\right) \sim -s\ln(s) + s - \frac{1}{2}\ln(s) + \Re(a)\ln(s)$$

as $s \to \infty$. Using these estimates, we obtain $\ln(G(s)) \sim (q-p)(s\ln(s)-s)$. Therefore, the integral exists if $p > q$. If $p = q$, then $\ln(G(s)x^{-s})$ will decrease exponentially for $|x| > 1$ due to x^{-s} decreasing exponentially.

The integral exists for $|x| = 1$ if the algebraic decay of $G(s)$ as $s \to \infty$ is sufficient. $\prod_{j=1}^{m} \Gamma(b_j + s)$ contributes

$$-\frac{1}{2}m + \sum_{j=1}^{m} \Re(b_j)$$

to the algebraic decay of $G(s)$, $\prod_{j=1}^{n} \Gamma(1 - a_j - s)$ contributes

$$-\frac{1}{2}n + \sum_{j=1}^{n} 1 - \Re(a_j),$$

$\frac{1}{\prod_{j=n+1}^{p} \Gamma(a_j+s)}$ contributes

$$\frac{1}{2}(p - n) - \sum_{j=n+1}^{p} \Re(a_j),$$

and $\frac{1}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s)}$ contributes

$$\frac{1}{2}n - \sum_{j=m+1}^{q} 1 - \Re(b_j).$$

Summing these terms yields

$$\sum_{i=1}^{q} \Re(b_j) - \sum_{i=1}^{p} \Re(a_j). \tag{3.2}$$

Therefore, the integral exists for $|x| = 1$ if expression (3.2) < -1.

Futhermore, the integral exists when C is the loop from $-\infty$ back to $-\infty$ if $q > p$, $|x| < 1$ or $|x| = 1$ and $\sum_{i=1}^{q} \Re(b_j) - \sum_{i=1}^{p} \Re(a_j) < -1$. Hence, we have the following:

1. C is a path from $-i\infty$ to $i\infty$ such that all poles of $\Gamma(b_j + s)$, $j = 1 \ldots, m$ are on the left and all poles of $\Gamma(1 - a_k - s)$, $k = 1 \ldots, n$ are on the right if $|\arg(x)| < \pi \left(n + m - \frac{p+q}{2}\right)$ or $|\arg(x)| = \pi \left(n + m - \frac{p+q}{2}\right)$ and expression (3.1) < -1 where σ is the real part of s as $s \to i\infty, -i\infty$.

2. C is a loop starting and ending at ∞ encircling all poles of $\Gamma(1 - a_k - s)$, $k = 1 \ldots, n$ exactly once in the negative direction and not encircling any poles of $\Gamma(b_j + s)$, $j = 1 \ldots, m$ if $p > q$ or $p = q$ and expression (3.2) < -1.

3. C is a loop starting and ending at $-\infty$ encircling all poles of $\Gamma(b_j + s)$, $j = 1 \ldots, m$ exactly once in the positive direction and not encircling any poles of $\Gamma(1 - a_k - s)$, $k = 1 \ldots, n$ if $q > p$ or $p = q$ and expression (3.2) < -1.

If more than one path can be chosen, the result of the integral will be the same for each path. This follows from Cauchy's residue theorem.

3.2 Definite Integration of a Single Meijer G Function

One of the important properties of Meijer G functions is its integration formulae. For integration of a single Meijer G function, the following equation can be used:

$$\int_0^\infty t^{s-1} G^{m,n}_{p,q}\left(tx \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dt = \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)} x^{-s}. \tag{3.3}$$

The conditions for which the formula is valid have been omitted and can be found in [4, page 347]. It is not a coincidence that the integrand of the Meijer G function definition appears on the right hand side of (3.3). This is because the left hand side of (3.3) is a Mellin transform and Meijer G functions are defined as a contour integral that looks very similar to an inverse Mellin transform. The formula is useful because if a function can be represented as a Meijer G function then (3.3) can be applied to compute the Mellin transform of the function.

3.3 Definite Integration of the Product of Two Meijer G Functions

By far, the most important property of Meijer G functions is that the definite integral from 0 to infinity of a product of two Meijer G functions can be represented by a single Meijer G function,

$$\int_0^\infty t^{s-1} G^{s,t}_{u,v}\left(\sigma t \,\middle|\, \begin{matrix} c_1,\dots,c_u \\ d_1,\dots,d_v \end{matrix}\right) G^{m,n}_{p,q}\left(\omega t \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dt =$$
$$\sigma^{-s} G^{m+t,n+s}_{p+v,q+u}\left(\frac{\omega}{\sigma} \,\middle|\, \begin{matrix} -d_1,\dots,-d_s,a_1,\dots,a_p,-d_{s+1},\dots,-d_u \\ -c_1,\dots,-c_t,b_1,\dots,b_q,-c_{t+1},\dots,-c_v \end{matrix}\right).$$

Again, the conditions under which the formula is valid have been omitted. Also, slightly more general formulae do exist. The conditions under which these formulas are valid can be found in [4, page 346].

The formula above produces a method for computing definite integrals of the form:

$$\int_0^\infty f(t)g(xt)dt,$$

where $f(x)$ and $g(x)$ are Meijer G functions. The first step is to find the Meijer G representation of the functions that are to be integrated, then apply the formula.

Finally it is usually useful to convert the resulting Meijer G function back into simpler functions, see [15]. The technique computes many of the integrals listed in the integration tables of [2], [3], [4], and [5]. Computer algebra systems also use this technique extensively to compute many different integrals.

We end this chapter with an example that demonstrates the use of Meijer G functions to compute a definite integral.

Example 3.3.1. Consider computing the integral

$$\int_0^\infty \operatorname{erf}(t) J_m(xt)dt.$$

Let $f(x) = \operatorname{erf}(x)$ and $g(x) = J_m(x)$. Using tables such as those found in [4] or a computer algebra system, we obtain the representations

$$f(x) = \frac{G_{1,2}^{1,1}\left(x^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)}{\sqrt{\pi}}$$

and

$$g(x) = G_{0,2}^{1,0}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right).$$

Therefore,

$$\int_0^\infty \operatorname{erf}(t) J_m(xt)dt = \frac{1}{\sqrt{\pi}} \int_0^\infty G_{1,1}^{1,2}\left(t^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) G_{0,2}^{1,0}\left(\frac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right)dt.$$

Let $u = t^2$ so $dt = \frac{1}{2\sqrt{u}}du$ and hence

$$\frac{1}{\sqrt{\pi}} \int_0^\infty G_{1,1}^{1,2}\left(t^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) G_{0,2}^{1,0}\left(\frac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right)dt = \\ \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}} G_{1,1}^{1,2}\left(u \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) G_{0,2}^{1,0}\left(\frac{1}{4}x^2u \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right)du.$$

The Meijer G function has the following property,

$$x^s G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} s + a_1, \ldots, s + a_p \\ s + b_1, \ldots, s + b_q \end{matrix}\right)$$

so that

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}} G_{1,1}^{1,2}\left(u \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) G_{0,2}^{1,0}\left(\frac{1}{4}x^2u \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right)du = \\ \frac{1}{2\sqrt{\pi}} \int_0^\infty G_{1,1}^{1,2}\left(u \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right) G_{0,2}^{1,0}\left(\frac{1}{4}x^2u \,\middle|\, \begin{matrix} \\ \frac{m}{2}, -\frac{m}{2} \end{matrix}\right)du.$$

Using the formula for the definite integration of two Meijer G functions gives

$$\frac{1}{2\sqrt{\pi}} G_{2,3}^{2,1}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ -\frac{1}{2}, \frac{m}{2}, -\frac{m}{2} \end{matrix}\right).$$

Finally, converting this Meijer G function to more standard functions using Roach's algorithm [15] gives the solution

$$\frac{\frac{\sqrt{\pi}\sec\left(\frac{1}{2}\pi m\right)\cos\left(\frac{1}{2}\pi m\right)}{\sqrt{x^2}} + \frac{\frac{1}{2}2^{-2m}\sqrt{\pi}\left(x^2\right)^{\frac{1}{2}m}{}_2F_2\left(\frac{1}{2}m+1,\frac{1}{2}m+\frac{1}{2};1+m,\frac{3}{2}+\frac{1}{2}m;-\frac{1}{4}x^2\right)}{\left(-\frac{1}{2}-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}m+\frac{1}{2}\right)}}{\sqrt{\pi}}. \tag{3.4}$$

We omit determining the region of validity of (3.4). □

Chapter 4

Asymptotics And Formal Solutions

4.1 Asymptotic Expansions

Asymptotic expansions (also known as asymptotic series or Poincaré expansions) are a formal series of functions such that when the series is truncated to a finite number of terms, the finite series is an approximation as the function tends to a particular point. The term formal implies that the convergence of the series is not considered. The results in this section are standard and can be found in almost any reference text concerning asymptotic expansions. The theorems from this section are found in [6] and [14].

Definition 4.1.1. Let $f(x)$ be a real valued function continuous in some region R and x_0 a point in R or a limit point of R. The formal power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{4.1}$$

is an asymptotic power series of $f(x)$ as $x \to x_0$ on R if

$$\lim_{x \to x_0} \left((x - x_0)^{-m} \left(f(x) - \sum_{n=0}^{m} a_n (x - x_0)^n \right) \right) = 0 \tag{4.2}$$

for all $m \in \mathbb{Z}_{\geq 0}$. □

It can be easily shown that (4.2) implies that for all $m \in \mathbb{Z}_{\geq 0}$

$$f(x) = \sum_{n=0}^{m} a_n (x - x_0)^n + O\left((x - x_0)^{m+1}\right) \tag{4.3}$$

as $x \to x_0$ in R. Note that neither (4.2) nor (4.3) imply in any way that the formal power series (4.1) converges to $f(x)$ or converges at all. Therefore, in general, we cannot write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Rather, we use the notation:

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

as $x \to x_0$ in R to imply that (4.2) holds and we do not know if the series converges to $f(x)$.

Example 4.1.1. We have already seen in Example 2.1.1 an asymptotic power series,

$$e^{-x} \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}. \tag{4.4}$$

Taylor's theorem states that if $f(x)$ is infinitely differentiable at a point $x = x_0$, then

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} \tag{4.5}$$

as $x \to x_0$ in the neighbourhood of the point x_0. In the case of e^{-x}, we can obtain (4.4) and show the series converges to obtain (2.1) in Example 2.1.1. However, Taylor's theorem alone does not imply that (4.5) converges to $f(x)$. Consider the following function,

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}. \tag{4.6}$$

One can show that $f(x)$ is infinitely differentiable at the point $x = 0$ and $f^{(n)}(0) = 0$. Therefore, by Taylor's theorem $f(x) \sim 0$. However, $f(x) = 0$ does not hold for $x > 0$. □

We can also consider an asymptotic power series as $x \to \infty$.

Definition 4.1.2. A function $f(x)$ is said to admit an asymptotic power series

$$\sum_{n=0}^{\infty} a_n x^{-n} \tag{4.7}$$

as $x \to \infty$ if

$$\lim_{x \to x_0} \left(x^m \left(f(x) - \sum_{n=0}^{m} a_n x^{-n} \right) \right) = 0 \tag{4.8}$$

for all $m \in \mathbb{Z}_{\geq 0}$. □

Similarly to the way (4.2) implies (4.3), we have (4.8) implies

$$f(x) = \sum_{n=0}^{m} a_n x^{-n} + O\left(x^{-m-1}\right). \tag{4.9}$$

If $f(x)$ satisfies (4.8), we use the same notation as before,

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

We do not know if the formal power series (4.7) converges to $f(x)$.

Next, we consider extending the idea of an asymptotic power series to general asymptotic series. An important property of the asymptotic power series of a function f is the difference between f and the Nth partial sum of the series is $O\left((x-x_0)^N\right)$ as $x \to x_0$. This is justifiable since as $x \to x_0$, $(x-x_0)^{m+1} = o\left((x-x_0)^m\right)$ as $x \to x_0$, where $f(x) = o\left(g(x)\right)$ as $x \to x_0$ holds if for all $\epsilon > 0$ there exists a neighborhood N_ϵ of x_0 such that $|f(x)| \leq \epsilon\,|g(x)|$ for all $x \in N_\epsilon \cap R$, R being the region under consideration. The small "o" notation should not be confused with the big "O" notation as they are not the same.

Definition 4.1.3. A sequence of continuous functions $\{\phi_n(x)\}$ is called an asymptotic sequence on R as $x \to x_0$ if:

1. $\phi_i(x)$ is continuous on R for all i,
2. x_0 is a point in R or a limit point of R, and
3. $\phi_{m+1}(x) = o\left(\phi_m(x)\right)$ as $x \to x_0$.

□

Asymptotic sequences of functions give rise to a generalization of asymptotic power series.

Definition 4.1.4. Let $f(x)$ be a real valued function, continuous in some region R, and let $\{\phi_n(x)\}$ be an asymptotic sequence as $x \to x_0$ on R. Then, the formal series

$$\sum_{n=0}^{\infty} a_n \phi_n(x) \tag{4.10}$$

is an asymptotic series of $f(x)$ as $x \to x_0$ on R with respect to $\{\phi_n(x)\}$ if

$$\lim_{x \to x_0} \left(\frac{1}{\phi_m(x)} \left(f(x) - \sum_{n=0}^{m} a_n \phi_n(x) \right) \right) = 0 \tag{4.11}$$

for all $m \in \mathbb{Z}_{\geq 0}$. □

Again, as (4.2) implies (4.3) and (4.8) implies (4.9), we also have (4.11) implies

$$f(x) = \sum_{n=0}^{m} a_n \phi_n(x) + O(\phi_{m+1}(x)),$$

as $x \to x_0$ on R. If $f(x)$ satisfies (4.11) then we again use the same notation as before,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x),$$

since we do not know if the formal power series (4.10) converges to $f(x)$.

An important property of these asymptotic expansions is that they are unique. This is justified by the following theorem.

Theorem 4.1.1. *Let*

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x)$$

be an asymptotic expansion of $f(x)$ *with respect to the asymptotic sequence* $\{\phi_n(x)\}$ *as* $x \to x_0$ *on some region* R*. Then the coefficients of* a_n *are uniquely determined.*

□

However the converse of Theorem 4.1.1, that is, an asymptotic expansion uniquely determines the function it comes from, is not true. This is illustrated in Example 4.1.1 where the function (4.6) has the same asymptotic expansion as zero as $x \to 0$ but (4.6) is not the same function as zero for any $x > 0$.

Example 4.1.2. Consider computing the asymptotic expansion of the function

$$f(x) = \sqrt{1 - \sin(x)}$$

as $x \to 0$. Suppose the asymptotic sequence used is the simple power series, $\{x^n\}$. Then

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$$

where

$$\begin{aligned}
a_0 &= \lim_{x\to 0} \sqrt{1-\sin(x)} \\
&= 1 \\
a_1 &= \lim_{x\to 0} \frac{\sqrt{1-\sin(x)}-1}{x} \\
&= \lim_{x\to 0} \frac{-cos(x)}{2\sqrt{1-sin(x)}} \text{ (L'Hôpital's rule)} \\
&= -\frac{1}{2} \\
a_2 &= \lim_{x\to 0} \frac{\sqrt{1-\sin(x)}-1+\frac{1}{2}x}{x^2} \\
&= \lim_{x\to 0} \left(\frac{-\cos^2(x)}{8(1-\sin(x))^{\frac{3}{2}}} + \frac{\sin(x)}{4\sqrt{1-\sin(x)}} \right) \text{ (L'Hôpital's rule twice)} \\
&= -\frac{1}{8}
\end{aligned}$$

and so forth to obtain

$$f(x) \sim 1 - \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} - \frac{x^6}{46080} + \frac{x^7}{645120} + O(x^8). \qquad (4.12)$$

We can also compute the asymptotic expansion of the function with respect to the asymptotic sequence $\{\sin^n(x)\}$. This is actually easier to determine because we can perform the substitution $y = \sin(x)$. As the expansion of

$$\sqrt{1-y}$$

with respect to $\{y^n\}$ is well known to be

$$\sqrt{1-y} \sim \sum_{n=0}^{\infty} \frac{(2n)!y^n}{(1-2n)(n!)^2 4^n},$$

we have,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(2n)!\sin^n(x)}{(1-2n)(n!)^2 4^n}.$$

We obtain the closed form for all of the coefficients of the expansion. Determining a closed form for (4.12) requires the following identities:

$$\begin{aligned}
\sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
1-\sin(x) &= 1-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&= \left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)^2 \\
\sqrt{1-\sin(x)} &= \left|\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right|.
\end{aligned}$$

Using the well known expansions about zero for $\sin(x)$ and $\cos(x)$, we obtain the closed form expansion:

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!}.$$

□

Up to this point, in this section we have only been considering functions defined on the real line. We are, however, also required to consider functions defined in the complex plane in later chapters. Fortunately, all the theory we have considered so far can be easily extended to the complex case. The only extra consideration we now must make is the valid directions we can take as $z \to z_0$. To do this, we specify a sector,

$$S_{\alpha\beta} = \{z | \alpha < \arg(z - z_0) < \beta\},$$

and change the definition of an asymptotic sequence as follows.

Definition 4.1.5. A sequence of functions $\{\phi_n(z)\}$ all continuous in a region R of the real line is called an asymptotic sequence as $z \to z_0$ in $S_{\alpha\beta}$, where z_0 is a point in R or a limit point of R, if

$$\phi_{m+1}(x) = o\left(\phi_m(x)\right)$$

holds uniformly as $z \to z_0$ in $S_{\alpha\beta}$. □

Then, an asymptotic series is extended in the following manner.

Definition 4.1.6. Let $f(z)$ be a complex valued function continuous in some region R and let $\{\phi_n(z)\}$ be an asymptotic sequence as $z \to z_0$ in $S_{\alpha\beta}$ on R. The formal series,

$$\sum_{n=0}^{\infty} a_n \phi_n(z), \tag{4.13}$$

is said to be an asymptotic series of $f(z)$ as $x \to x_0$ in $S_{\alpha\beta}$ on R with respect to $\{\phi_n(x)\}$ if

$$f(z) = \sum_{n=0}^{m} a_n \phi_n(z) + o\left(\phi_m(z)\right)$$

holds uniformly as $z \to z_0$ in $S_{\alpha\beta}$ for all $m \in \mathbb{Z}_{\geq 0}$. □

Note that the important uniqueness theorem still holds for these types of expansions.

4.2 Formal Solutions Of Differential Equations

A formal solution of a differential equation is a function expressed as an infinite sum that satisfies the differential equation. The solution is formal because we do not consider the convergence of the sum. We will only be concerned with formal solutions of linear homogeneous differential equations with polynomial coefficients. Recall that solutions of linear homogeneous differential equations with polynomial coefficients are holonomic functions. The theory of formal solutions are found in many texts pertaining to solutions of ordinary linear differential equations. The theorems below are found in [1] and are converted from matrix differential notation to a single equation notation.

Consider the following example of a formal solution that does not converge.

Example 4.2.1. The sum

$$\sum_{n=0}^{\infty} n! x^n$$

satisfies the differential equation

$$x^2 y''(x) + (3x - 1) y'(x) + y(x) = 0,$$

as a formal sum. However, by the Ratio test, the series does not converge. □

When dealing with linear homogeneous differential equations with polynomial coefficients, it is quite difficult to describe their solutions at all points. However, there is a general theory on describing all the formal solutions of the linear homogeneous differential equations with polynomial coefficients. The rest of this section will be dedicated to this theory.

When discussing formal solutions of linear homogeneous differential equations with polynomial coefficients at a point x_0, we can classify x_0 as one of three types of points: ordinary, regular singular, or irregular singular.

Definition 4.2.1. Suppose a linear homogeneous differential equation with polynomial coefficients is given as

$$\sum_{i=0}^{n} p_i(x) y^{(n)}(x) = 0. \tag{4.14}$$

1. An ordinary point, x_0 of (4.14) is a point at which $p_n(x_0)$ is not 0.
2. A point x_0 is a regular singular point if it is not an ordinary point and

   $$\frac{p_i(x)}{p_n(x)}$$

   has a pole of order at most i at x_0 for i from 1 to $n-1$.
3. An irregular singular point is simply a point that is not an ordinary point or a regular singular point.

□

The formal solutions of linear homogeneous differential equations with polynomial coefficients at an ordinary point are classified in the following theorem.

Theorem 4.2.1. *There exists formal solutions of a linear homogeneous differential equation with polynomial coefficients at an ordinary point x_0 and all solutions are power series about x_0. The set of solutions also form a vector space of dimension equal to the order of the differential equation. All power series have a radius of convergence that is a distance at least from x_0 to the closest point that is either a regular or irregular singularity.* □

Like ordinary points, we can classify formal solutions at regular singular points and find that they converge as well.

Theorem 4.2.2. *There exists formal solutions of a linear homogeneous differential equation with polynomial coefficients at a regular singular point x_0. Each solution can be written as a finite sum of terms in the form:*

$$\sum_{i=-k}^{\infty} \sum_{j=0}^{n} c_{i,j} \left((x - x_0)^{\frac{1}{r}} \right)^i (\log(x - x_0))^j,$$

(called a formal logarithmic sum) where n, r, k are positive integers and the $c_{i,j}$ are complex numbers. The set of solutions form a vector space of dimension equal to the order of the differential equation. All series have a radius of convergence that is a distance at least from x_0 to the closest point that is either a regular or irregular singularity (different from x_0). □

Note that in the previous two theorems, x_0 could be infinity in which case $x - x_0$ would be replaced with $\frac{1}{x}$.

Unlike ordinary points and regular singular points, formal solutions of linear homogeneous differential equations with polynomial coefficients at irregular singular points do not always correspond to actual solutions. That is, the formal solutions do not always converge. However, we can still describe the formal solutions themselves.

Theorem 4.2.3. *There exists formal solutions of a linear homogeneous differential equation with polynomial coefficients at an irregular singular point x_0. Each solution can be written as a finite sum of terms in the form:*

$$\exp\left(p\left((x-x_0)^{\frac{-1}{r}}\right)\right)\sum_{i=-k}^{\infty}\sum_{j=0}^{n}c_{i,j}\left((x-x_0)^{\frac{1}{r}}\right)^{i}(\log(x-x_0))^{j},$$

(called a formal exp-log sum) where n, r, k are positive integers, $p(y)$ is a polynomial in y and the $c_{i,j}$ are complex numbers. The set of solutions form a vector space of dimension that is equal to the order of the differential equation. □

Similar to Theorem 4.2.2 in the previous theorem, x_0 can be infinity in which case, $x - x_0$ would need to be replaced with $\frac{1}{x}$. Note that nothing can be said about the convergence of the formal solutions at irregular singular points. However, we have the following theorem.

Theorem 4.2.4. *For every formal solution of a linear homogeneous differential equation with polynomial coefficients at an irregular singular point point x_0, there exists an actual solution such that the actual solution has an asymptotic expansion that is equal to the formal solution as $x \to x_0$.* □

Since formal solutions of a linear homogeneous differential equation with polynomial coefficients converge at ordinary and regular singular points, they are also asymptotic expansions. Therefore, the asymptotic expansion of a solution of a linear homogeneous differential equation with polynomial coefficients can always be written as a finite sum of formal exp-log sums because a power series is a logarithmic sum, which is also an exp-log sum.

4.3 Analytic Continuation Of Mellin Transforms

In the previous sections and chapters, we have discussed Mellin transforms, asymptotic expansions, holonomic functions and formal solutions of linear homogeneous differential equations with polynomial coefficients. In this section, we will begin to tie these ideas together. The analytic continuation of Mellin transforms plays an important role in the methods presented in later chapters. In general, it is not obvious how one analytically continues a Mellin transform but some assumptions can be made. We assume the functions in the integrand of

$$I(x) = \int_0^\infty f(t)g(xt)dt$$

are holonomic functions. It follows that their asymptotic expansions can be written as a finite sum of exp-log sums. This assumption allows us to continue the Mellin transform of $f(x)$ and $g(x)$ to the entire complex plane. The results from this section are presented in [6].

Suppose $f(x)$ is a function and we are required to compute the analytic continuation of its Mellin transform. Assume the Mellin transform exists in the normal sense and $f(x)$ has a fundamental strip $\langle \alpha, \beta \rangle$ with $\alpha < \beta$. We will also assume $f(x)$ is holonomic and therefore $f(x)$ has an asymptotic expansion as $x \to \infty$ of the form:

$$f(x) \sim \exp\left(p\left(x^{\frac{1}{r}}\right)\right) \sum_{i=-k}^{\infty} \sum_{j=0}^{n} c_{i,j} \left(x^{\frac{-1}{r}}\right)^i (\log(x))^j \tag{4.15}$$

where n, r, k are positive integers, $p(y)$ is a polynomial in y and the $c_{i,j}$ are complex numbers. In general, $f(x)$ can be a finite sum of the above expansions. However without loss of generality, we assume (4.15) since the Mellin transform is a linear operator. We will also assume that

$$\lim_{x\to\infty} \Re\left(p\left(x^{\frac{1}{r}}\right)\right) < \infty$$

or the Mellin transform of $f(x)$ will not exist.

First, we continue the Mellin transform of $f(x)$, $\mathcal{M}[f(x); s]$ to the entire right half of the complex plane using the above assumptions. Three different cases based on $p(x^{\frac{1}{r}})$ arise. The first case occurs if $f(x)$ decreases exponentially at infinity.

Theorem 4.3.1. *If*

$$\lim_{x\to\infty} \Re\left(p\left(x^{\frac{1}{r}}\right)\right) = -\infty$$

then $f(x)$ decreases exponentially at infinity and therefore $\mathcal{M}[f(x); s]$ is holomorphic for $\Re(s) > \alpha$ and $\beta = \infty$. □

Next, we consider the case when $f(x)$ is oscillatory at infinity.

Theorem 4.3.2. *If*

$$\lim_{x\to\infty} \Re\left(p\left(x^{\frac{1}{r}}\right)\right)$$

is finite but

$$\lim_{x\to\infty} \Im\left(p\left(x^{\frac{1}{r}}\right)\right) = \pm\infty$$

then $f(x)$ is oscillatory at infinity and $\mathcal{M}[f(x);s]$ can be continued into $\Re(s) \geq \beta$ as a holomorphic function. $\square$

Finally, we consider the case when $f(x)$ has algebraic growth at infinity.

Theorem 4.3.3. *If*

$$\lim_{x\to\infty} \Re\left(p\left(x^{\frac{1}{r}}\right)\right)$$

is finite and

$$\lim_{x\to\infty} \Im\left(p\left(x^{\frac{1}{r}}\right)\right)$$

is finite, then $f(x)$ is algebraic at infinity and $\mathcal{M}[f(x);s]$ can be continued into $\Re(s) \geq \beta$ as a meromorphic function with poles at $\frac{-i}{r}$, $i = 0, 1, \ldots$ (assuming $c_{i,j} \neq 0$ for at least one $j = 0, 1, \ldots, n$). At each pole $\frac{-i}{r}$, $\mathcal{M}[f(x);s]$ has a Laurent expansion with singular part

$$\sum_{j=0}^{n} \frac{(-1)^{j+1} c_{i,j} i!}{(s - \frac{i}{r})^{j+1}}.$$

$\square$

In a similar process, we can continue $\mathcal{M}[f(x);s]$ to the entire left half of the complex plane by assuming $f(x)$ has an expansion of the form:

$$f(x) \sim \exp\left(q\left(x^{\frac{-1}{t}}\right)\right) \sum_{i=-l}^{\infty} \sum_{j=0}^{m} d_{i,j} \left(x^{\frac{1}{r}}\right)^{i} (\log(x))^{j} \tag{4.16}$$

where m, t, l are positive integers, $q(y)$ is a polynomial in y and the $d_{i,j}$ are complex numbers. We will also assume that

$$\lim_{x\to\infty} \Re\left(q\left(x^{\frac{-1}{t}}\right)\right) < \infty$$

since otherwise, the Mellin transform of $f(x)$ will not exist. Again, we have three cases analogous to the three cases above.

Theorem 4.3.4. *If*

$$\lim_{x\to\infty} \Re\left(q\left(x^{\frac{-1}{t}}\right)\right) = -\infty$$

then $f(x)$ decreases exponentially at zero and therefore $\mathcal{M}[f(x);s]$ is holomorphic for $\Re(s) < \beta$ and $\alpha = -\infty$. $\square$

Theorem 4.3.5. *If*

$$\lim_{x\to\infty} \Re\left(q\left(x^{\frac{-1}{t}}\right)\right)$$

is finite but

$$\lim_{x\to\infty} \Im\left(q\left(x^{\frac{-1}{t}}\right)\right) = \pm\infty,$$

then $f(x)$ is oscillatory at zero and $\mathcal{M}[f(x);s]$ can be continued into $\Re(s) \leq \alpha$ as a holomorphic function. □

Theorem 4.3.6. *If*

$$\lim_{x\to\infty} \Re\left(q\left(x^{\frac{-1}{t}}\right)\right)$$

is finite and

$$\lim_{x\to\infty} \Im\left(q\left(x^{\frac{-1}{t}}\right)\right)$$

is finite, then $f(x)$ is algebraic at zero and $\mathcal{M}[f(x);s]$ can be continued into $\Re(s) \leq \alpha$ as a meromorphic function with poles at $\frac{-i}{r}$, $i = 0, 1, \ldots$ (assuming $d_{i,j} \neq 0$ for at least one $j = 0, 1, \ldots, m$). At each pole $\frac{-i}{r}$, $\mathcal{M}[f(x);s]$ has a Laurent expansion with singular part

$$\sum_{j=0}^{M(i)} \frac{(-1)^{j+1} d_{i,j} i!}{(s + \frac{i}{t})^{j+1}}.$$

□

The next example shows the relationship between analytically continued Mellin transforms and asymptotic expansions.

Example 4.3.1. Consider the function e^{-x}. We have seen in Example 2.1.1 that the Mellin transform of e^{-x} is $\Gamma(s)$ and its fundamental strip is $\langle 0, \infty \rangle$. To determine the poles of the analytic continuation of $\Gamma(s)$, we only need to observe the asymptotic expansion of e^{-x} at 0 (since we do not need to continue $\Gamma(s)$ to the right part of the complex plane). We have also seen in Example 2.1.1 that

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

Therefore, by Theorem 4.3.3, $\Gamma(s)$ has a simple pole at every negative integer $-n$ and has a residue of $\frac{(-1)^n}{n!}$.

This theory can also be used in the opposite direction. Suppose we wish to find asymptotic expansions of the error function at zero and infinity. We can look up the Mellin transform of the error function to find that it is

$$-\frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}s\right)}{\sqrt{\pi}s}$$

with fundamental strip $\langle -1, 0 \rangle$. We have just determined the poles' location in the analytic continuation of $\Gamma(s)$. Thus, we use the result to determine the poles of the analytic continuation of the error function. To find an asymptotic expansion of the error function at zero, we simply sum up all the residues of its Mellin transform and multiply x^p at points p such that $\Re(p) \leq -1$. Doing so gives

$$\text{erf}(x) \sim \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i x^{1+2i}}{(1+2i)i!}$$

as $x \to 0$. We can check that this sum converges for all x, and that it also converges to $\text{erf}(x)$ for all x. Using a similar method, we obtain

$$\text{erf}(x) \sim 1$$

as $x \to \infty$. This converges but it doesn't converge to $\text{erf}(x)$ as $-1 < \text{erf}(x) < 1$ for all x. However, for large x, the error function is actually very close to 1. □

4.4 Asymptotic Expansions Of Integrals

Using the theory of Mellin transforms and asymptotic expansions outlined in previous sections of this chapter, we will present a process to compute the asymptotic expansion of the integral

$$I(x) = \int_0^\infty f(t)g(xt)dt$$

as $x \to \infty$. Suppose that both $f(x)$ and $g(x)$ are in $\mathcal{L}_{loc}(0,\infty)$ and the Mellin transforms of $f(x)$ and $g(x)$ exist. Let the fundamental strip of $f(x)$ be $\langle \alpha, \beta \rangle$ and the fundamental strip of $g(x)$ be $\langle \gamma, \delta \rangle$. Assume that the intersection of the strips $\langle 1-\beta, 1-\alpha \rangle$ and $\langle \gamma, \delta \rangle$ is not empty. Let $f(x)$ have an asymptotic expansion that is the sum of a finite number of terms as in (4.16) and $g(x)$ have an asymptotic expansion that is the sum of a finite number of terms as in (4.15). The details of the techniques presented in this chapter can be found in [6] and [14].

Since $f(x)$ and $g(x)$ are in $\mathcal{L}_{loc}(0,\infty)$ and the strips $\langle 1-\beta, 1-\alpha \rangle$ and $\langle \gamma, \delta \rangle$ overlap, Theorem 2.1.1 holds. Thus,

$$\mathcal{M}[I(x); s] = \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$$

and the fundamental strip of $I(x)$ is $\langle \max\{1-\beta\,\gamma\}, \min\{1-\alpha, \delta\} \rangle$. By Theorem 2.1.2, for any k in $(\max\{1-\beta\,\gamma\}, \min\{1-\alpha, \delta\})$, we have

$$I(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \phi(s)x^{-s}ds = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s}ds. \tag{4.17}$$

By the results in Section 4.3, we conclude that it is possible to analytically continue $\mathcal{M}[f(x); 1-s]$ to the right half of the complex plane (using the asymptotic expansion of $f(x)$ at zero). Similarly, it is also possible to analytically continue $\mathcal{M}[g(x); s]$ to the right half of the complex plane (using the expansion of $g(x)$ at infinity). This gives an analytic continuation of $\phi(s)$ in (4.17) to the right half of the complex plane. Also the theorems in Section 4.3 tell us where the poles of the continuations of $\mathcal{M}[f(x); 1-s]$ and $\mathcal{M}[g(x); s]$ are located as well as giving us the singular parts of the Laurent expansions at each of the poles.

Now, suppose that

$$\phi(K + iy) \in \mathcal{L}(-\infty < y < \infty) \text{ and} \tag{4.18}$$

$$\lim_{|y|\to\infty} \phi(x + iy) = 0 \text{ for all } x \in [k, K], \tag{4.19}$$

and no poles of $\phi(s)$ lie on the line $K - i\infty$ to $K + i\infty$. Let C be the contour from $k - it$ to $k + it$ to $K + it$ to $K - i\infty$ back to $k - it$. By Cauchy's Residue Theorem,

$$\lim_{t\to\infty} \frac{1}{2\pi i} \int_C \phi(s)x^{-s}ds = - \sum_{k<\Re(s)<K} \text{res}(\phi(s)x^{-s}). \tag{4.20}$$

However, since

$$\lim_{|y|\to\infty} \phi(R + iy) = 0$$

for all $x \in [k, K]$ the parts of contour C from (K, it) to $(K, -it)$ and $(K, -it)$ to $(k, -it)$ contribute nothing to (4.20). Thus,

$$\frac{1}{2\pi i} \lim_{t\to\infty} \int_{k-it}^{k+it} \phi(s)x^{-s}ds = - \sum_{k<\Re(s)<K} \text{res}(\phi(s)x^{-s}) + \frac{1}{2\pi i} \lim_{t\to\infty} \int_{K-it}^{K+it} \phi(s)x^{-s}ds.$$

Since $\phi(K + iy)$ is in $\mathcal{L}(-\infty < y < \infty)$, we have

$$\left| \lim_{t\to\infty} \int_{K-it}^{K+it} \phi(s)x^{-s}ds \right| \le x^{-K} \int_{-\infty}^{\infty} \left|\phi(s)x^{-s}ds\right| = O(x^{-K})$$

which implies

$$\frac{1}{2\pi i} \lim_{t\to\infty} \int_{k-it}^{k+it} \phi(s)x^{-s}ds \sim - \sum_{k<\Re(s)<K} \text{res}(\phi(s)x^{-s}) + O(x^{-K}).$$

Rewriting the above using (4.17) results in

$$I(x) \sim - \sum_{k<\Re(s)<K} \text{res}(\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s}) + O(x^{-K}),$$

a finite asymptotic expansion of $I(x)$. If this is true for arbitrary large K, then

$$I(x) \sim - \sum_{k<\Re(s)<\infty} \text{res}(\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s})$$

is an infinite asymptotic expansion of $I(x)$.

Using the theory of residues and knowledge of the singular parts of the Laurent expansions of $\mathcal{M}[f(x); 1-s]$ and $\mathcal{M}[g(x); s]$ in the right half of the complex plane, we can often expand the residue sums in more explicit terms. For example, if $\mathcal{M}[f(x); 1-s]$ and $\mathcal{M}[g(x); s]$ have no poles in the right half of the complex plane, then $I(x) \sim 0$. If $\mathcal{M}[f(x); 1-s]$ has no poles in the right half of the complex plane, then the expansion of $I(x)$ depends only on the singular parts of the Laurent expansions at the poles of $\mathcal{M}[g(x); s]$ and the function $\mathcal{M}[f(x); 1-s]$. Note that in this case, we are not required to know $\mathcal{M}[g(x); s]$ explicitly. Only the expansion of $g(x)$ at infinity is needed. It is important to consider these special cases, since as illustrated in the last two examples, it is sometimes possible to avoid computing one or even both Mellin transforms.

Similarly to the above, we can also compute the expansion of $I(x)$ as $x \to 0$. Instead of shifting the contour of integration in (4.17) to the right, we shift it to the left to obtain

$$I(x) \sim \sum_{K<\Re(s)<k} \operatorname{res}(\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s}) + O(x^K).$$

If the proper conditions are met for arbitrarily large K, then we obtain the infinite expansion

$$I(x) \sim \sum_{-\infty<\Re(s)<k} \operatorname{res}(\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]x^{-s}).$$

Here, the proper conditions are that $\phi(K+iy)$ is in $\mathcal{L}(-\infty < y < \infty)$ and

$$\lim_{|y|\to\infty} \phi(x+iy) = 0$$

for all $x \in [K, k]$. When computing the expansion as x tends to zero, we must continue $\mathcal{M}[f(x); 1-s]$ and $\mathcal{M}[g(x); s]$ to the left half of the complex plane. We assume $f(x)$ and $g(x)$ have proper exp-log sums as expansions as $x \to \infty$ and $x \to 0$ respectively.

Finally let us return to Example 2.1.4 and compute $I(x)$.

Example 4.4.1. In Example 2.1.4 we attempted to determine

$$I(x) = \int_0^\infty f(t)\sin(xt)dt,$$

and found that

$$I(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} x^{-s}\mathcal{M}[f(x); 1-s]\left(2^{-1+s}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(-\frac{1}{2}s+1)}\right) ds,$$

with $k \in (1-\beta, 1-\alpha) \cap (-1, 0) \neq \emptyset$, $\langle \alpha, \beta \rangle$ the fundamental strip of $f(x)$. Assume that the Mellin transform of $f(x)$ exists and

$$f(x) \sim \sum_{i=0}^{\infty} c_i x^{r_i}$$

as $x \to 0$ where all $r_i > 1$. If this is true, then $(1-\beta, 1-\alpha) \cap (-1, 0) \neq \emptyset$ and the only contributing poles are those of $\mathcal{M}[f(x); 1-s]$ in the expansion of $I(x)$ as $x \to \infty$ (since $\sin(x)$ is an oscillating function and has no poles in the right half of the complex plane). It follows that

$$I(x) \sim \sum_{i=0}^{\infty} c_i \left(2^{-r_i} \sqrt{\pi} \frac{\Gamma(1 - \frac{1}{2} r_i)}{\Gamma(\frac{1}{2} r_i + \frac{1}{2})} \right) x^{r_i - 1}$$

assuming the assumptions (4.18) and (4.19) hold for arbitrarily large K. □

Chapter 5

The Holonomic Mellin Convolution Algorithm

In this chapter we will present an algorithm that will compute a differential equation that annihilates

$$I(x) = \int_0^\infty f(t)g(xt)dt$$

for $f(x)$ and $g(x)$ homonomic. This differential equation will be of the form

$$\sum_{i=0}^{n} p_i(x)I^{(i)}(x) = h(x) \tag{5.1}$$

where the $p_i(x)$ are polynomials and $h(x)$ will be of a slightly more general form than a polynomial. From the special form of $h(x)$, (5.1) can be converted into a linear homogeneous differential equation with polynomial coefficients, albeit, possibly with much larger order.

We will assume that both $f(x)$ and $g(x)$ are holonomic functions so that they both satisfy linear homogeneous differential equations with polynomial coefficients. From these differential equations, we compute recurrence equations that annihilate $\mathcal{M}[f(x); 1-s]$ and $\mathcal{M}[g(x); s]$. These recurrence equations are used to compute a recurrence equation that annihilates $\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$ which, under certain assumptions, is equal to $\mathcal{M}[I(x); s]$. We will then show how to recover a differential equation (in the form of (5.1)) that is satisfied by $I(x)$. To make this possible, we assume there is an oracle that can compute any number of leading terms of the exp-log sum expansions of holonomic functions and another oracle that computes residues. The theory of Mellin transforms guarantees, when certain conditions are met, that the algorithm returns a differential equation that is satisfied by $I(x)$ for x along the positive real line and that the integral also converges along that line.

In the last section of this chapter, we will discuss the procedure to obtain an explicit closed form solution of the integral. This is performed by finding a general solution of the differential equation satisfied by $I(x)$. We compute leading terms of series that converge to the integral in certain regions. These series are considered local solutions of the differential equation around a point. Solving a system of linear equations equates one particular solution of the differential equation with the local solution that converges to the integral. The integral equals this particular solution in the region where converegence of the local solutions occurs. This is performed for points zero and infinity, and combining the resulting particular solutions and their respective regions of validity, gives an explicit closed form solution of the integral.

5.1 Annihilating Mellin Transforms

In this section, we present an algorithm that outputs a recurrence equation with polynomial coefficients that annihilates $\mathcal{M}[g(x); s]$. The input to the algorithm is a linear homogeneous differential equation with polynomial coefficients satisfied by $g(x)$. We will assume $g(x)$ has a non-empty fundamental strip. By the results in Sections 4.2 and 4.3, we continue the Mellin transform of $\mathcal{M}[g(x); s]$ to the entire complex plane as a meromorphic function.

Using Theorems 2.1.6 and 2.1.7 we obtain an analytic continuation of $\mathcal{M}[x^j g^{(i)}(x); s]$ to the entire complex plane as

$$\mathcal{M}[x^j g^{(i)}(x); s] = \mathcal{M}[g^{(i)}(x); s + j] = (-1)^i \frac{\Gamma(s + j)}{\Gamma(s + j - i)} \mathcal{M}[g(x); s + j - i]$$

for all non-negative integers i and j. Writing the linear homogeneous differential equation for $g(x)$ as

$$\sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} x^j g^{(i)}(x) = 0$$

and then applying the Mellin transform, we obtain

$$\begin{aligned}
\mathcal{M}\left[\sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} x^j g^{(i)}(x); s\right] &= \mathcal{M}[0; s], \\
\sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} \mathcal{M}\left[x^j g^{(i)}(x); s\right] &= 0, \\
\sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} (-1)^i \frac{\Gamma(s + j)}{\Gamma(s + j - i)} \mathcal{M}[g(x); s + j - i] &= 0. \qquad (5.2)
\end{aligned}$$

Therefore (5.2) is a recurrence relation that annihilates the analytic continuation of $\mathcal{M}[g(x); s]$. Since i and j are non-negative integers,

$$\frac{\Gamma(s + j)}{\Gamma(s + j - i)}$$

is always a polynomial in s.

Algorithm 1: DEtoMellinRE

Input: A differential equation $g(x)$ satisfies, $\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}x^j g^{(i)}(x) = 0$.

Output: A recurrence relation with polynomial coefficients that annihilates the analytic continuation of $\mathcal{M}[g(x); s]$.

begin

  return $\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}(-1)^i \frac{\Gamma(s+j)}{\Gamma(s+j-i)} u(s+j-i) = 0$

end

In order to compute the recurrence relation that annihilates $\mathcal{M}[f(x); 1-s]$, we need to transform the differential equation satisfied by $f(x)$. By Theorem 2.1.6

$$\mathcal{M}[f(x); 1-s] = \mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right); s\right],$$

assuming $f(x)$ has a non-empty fundametal strip. Therefore, if a differential equation is computed which is satisfied by $\frac{1}{x} f\left(\frac{1}{x}\right)$, then we apply DEtoMellinRE to that differential equation to obtain a recurrence relation that annihilates $\mathcal{M}[f(x); 1-s]$. By Theorem 2.2.1, since $f(x)$ is holonomic, $\frac{1}{x} f\left(\frac{1}{x}\right)$ is also holonomic and we can compute the desired differential equation.

The next step of the process outlined at the beginning of this chapter is to compute a recurrence equation that annihilates the Mellin transform of $I(x)$. As was mentioned previously, we assume there is an oracle that computes any number of leading terms of the exp-log sum expansions of holonomic functions. Thus, we can compute the fundamental strips $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ of $f(x)$ and $g(x)$, respectively. To do this, we apply Theorem 2.1.1 and observe the leading term of $f(x)$ and $g(x)$ at zero and infinity. Once the fundamental strips are computed, we verify whether they are non-empty and whether they overlap. If the strips are non-empty and overlap then by Theorem 2.1.1, we have

$$\mathcal{M}[I(x); s] = \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$$

and $I(x)$ has a fundamental strip $\langle \max\{1-\beta, \gamma\}, \min\{1-\alpha, \delta\}\rangle$. We use Theorem 2.2.1 to compute the recurrence equation that annihilates their product $\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$. This annihilates $\mathcal{M}[I(x); s]$.

Example 5.1.1. Suppose one wishes to find a differential equation that is satified by

$$I(x) = \int_0^\infty K_0(t)\sin(xt)dt.$$

The first step is to compute the recurrence equation that annihilates the Mellin transform of $I(x)$. Note that $f(x) = K_0(x)$ and $g(x) = \sin(x)$ are both holonomic with

$$xf''(x) + f'(x) - xf(x) = 0 \text{ and } g''(x) + g(x) = 0.$$

Using the closure properties of holonomic functions we find that $\hat{f}(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ is holonomic and satisfies

$$x^4 \hat{f}''(x) + 3x^3 hatf'(x) + (x^2 - 1)\hat{f}(x) = 0.$$

Computing the leading terms of expansions of $f(x)$ and $g(x)$ gives

$$\begin{aligned} f(x) &\sim O(-\ln(x)) \text{ as } x \to 0, \\ f(x) &\sim O\left(\frac{1}{2e^x}\sqrt{\frac{2\pi}{x}}\right) \text{ as } x \to \infty, \\ g(x) &\sim O(x) \text{ as } x \to 0 \text{ and} \\ g(x) &\sim O\left(-\frac{1}{2}ie^{ix} + \frac{1}{2}ie^{-ix}\right) \text{ as } x \to \infty. \end{aligned}$$

This implies that $f(x)$ has a fundamental strip of $\langle 0, \infty\rangle$ and $g(x)$ has a fundamental strip of $\langle -1, 0\rangle$. Therefore $\mathcal{M}[I(x); s]$ exists and $I(x)$ has fundamental strip $\langle -1, 0\rangle$. Running DEtoMellinRE on the differential equations yields

$$(s^2 + 2s + 1)u(s + 2) - u(s) = 0 \text{ and } v(s + 2) + (s^2 + s)v(s) = 0$$

which annihilate $\mathcal{M}[f(x); 1 - s]$ and $\mathcal{M}[g(x); s]$, respectively. Using holonomic closure properties we then find that $\mathcal{M}[I(x); s]$ satisfies

$$(s + 1)w(s + 2) + sw(s) = 0. \tag{5.3}$$

The next section is devoted to determing how to obtain the differential equation satisfied by $I(x)$ from (5.3). □

5.2 Recovering the Differential Equation

In the previous section we described a procedure that computes a recurrence equation to annihilate the Mellin transform of a holonomic function. This section develops an algorithm that returns a differential equation satisfied by a function given a recurrence equation that annihilates the Mellin transform of the function. We will also need the ability to compute the leading terms of the expansion of the function at zero and infinity. Note that the differential equation returned may not be homogeneous, but will be linear and have polynomial coefficients. The algorithm recovers the differential equation satisfied by $I(x)$ since we can compute the leading terms of the expansion of $I(x)$ using theory from Section 4.4.

The recovery algorithm is based on the following theorem.

Theorem 5.2.1. *Let $I(x)$ be a function with a fundamental strip $\langle \alpha, \beta \rangle$ where $\alpha < \beta$. Suppose k is such that $\alpha < k < \beta$. Assume*

$$\lim_{|\Im(s)|\to\infty} s^j \mathcal{M}[I(x); s] = 0 \text{ for all } s \text{ such that } \Re(s) \in [k, k+c] \tag{5.4}$$

and the line $\Re(s) = k + c$ contains no poles of $\mathcal{M}[I(x); s]$. Then

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} (s+c)^j \mathcal{M}[I(x); s+c] x^{-s} ds \\ &\quad - \sum_{k<\Re(s)<k+c} Res(x^{-s+c} s^j \mathcal{M}[I(x); s]). \end{aligned}$$

Proof. By Theorems 2.1.6 and 2.1.7,

$$\mathcal{M}\left[\left(-x\frac{d}{dx}\right)^j I(x); s\right] = s^j \mathcal{M}[I(x); s]$$

with $\left(-x\frac{d}{dx}\right)^j I(x)$ having a fundamental strip of $\langle \alpha, \beta \rangle$. Therefore by Theorem 2.1.2,

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= x^c \int_{k-i\infty}^{k+i\infty} s^j \mathcal{M}[I(x); s] x^{-s} ds \\ &= \int_{k-i\infty}^{k+i\infty} s^j \mathcal{M}[I(x); s] x^{-s+c} ds. \end{aligned}$$

Let C be the contour from $k - iu$ to $k + iu$ to $k + c + iu$ to $k + c - iu$ back to $k - iu$ and consider

$$\lim_{u\to\infty} \int_C s^j \mathcal{M}[I(x); s] x^{-s+c} ds.$$

By (5.4), the parts of the contour from $k + iu$ to $k + c + iu$ and from $k + c - iu$ back to $k - iu$ contribute nothing to the integral as $u \to \infty$. Therefore, by Cauchy's Residue Theorem, we have

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= \lim_{u\to\infty} \int_{k-iu}^{k+iu} s^j \mathcal{M}[I(x); s] x^{-s+c} ds \\ &= \lim_{u\to\infty} \int_{k+c-iu}^{k+c+iu} s^j \mathcal{M}[I(x); s] x^{-s+c} ds \\ &\quad - \sum_{k<\Re(s)<k+c} \text{Res}(x^{-s+c} s^j \mathcal{M}[I(x); s]). \end{aligned}$$

Substituting $s' = s - c$ into the last integral gives

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= \lim_{u\to\infty} \int_{k-iu}^{k+iu} (s'+c)^j \mathcal{M}[I(x); s'+c] x^{-s'} ds' \\ &\quad - \sum_{k<\Re(s)<k+c} \text{Res}(x^{-s+c} s^j \mathcal{M}[I(x); s]). \end{aligned}$$

□

Theorem 5.2.1 provides a method to perform the inverse Mellin transform on a portion of a recurrence relation that annihilates a Mellin transform. If the inversion theorem can be applied systematically, the inverse Mellin transform on the entire recurrence equation can be obtained. This results in a linear differential equation with polynomial coefficients with a homogeneous term that is the sum of residues of the form $\text{Res}(x^{-s+c} s^j \mathcal{M}[h(x); s])$ where c and j are integers. A systematic approach to performing the inverse Mellin transforms is given by the algorithm MellinREtoDE.

Theorem 5.2.2. *Algorithm MellinREtoDE is correct.*

Proof. Consider ordering the coefficients $c_{i,j}$. We order $c_{i,j}$ before $c_{i',j'}$ if $i > i'$ or $i = i'$ and $j > j'$. The For loops in MellinREtoDE are structured to loop through coefficients in order starting with the coefficient that precedes all others. For each coefficient, the algorithm checks whether Theorem 5.2.1 can be applied with $c = a$ and $j = b$. When possible, the algorithm adds the parts specified by Theorem 5.2.1 to the differential equation (DE) and the homogeneous term ($HTerm$). It then updates the recurrence relation (Rec) subtracting the portion of the recurrence with the inverse Mellin transform applied to it according to Theorem 5.2.1. After this subtraction, the coefficient considered becomes zero. Only coefficients of the recurrence that follow the one in consideration are modified. Therefore, after the loops have completed, Rec will be zero. Thus, the inverse Mellin transform has been applied to the entire recurrence relation and the resulting differential equation is satisfied by $I(x)$. □

In the previous section, we obtained a recurrence relation that annihilated $\mathcal{M}[I(x); s]$. MellinREtoDE can be applied to this recurrence equation to obtain a differential equation that is satisfied by $I(x)$. The differential equation may have a homogeneous term made up of a finite sum with terms of the form

$$c \sum_{k<\Re(s)<k+a} \text{Res}(x^{-s+a} s^b \mathcal{M}[I(x); s]),$$

where c is some known constant, k is a real value inside the fundamental strip of $I(x)$, and a, b are known integers. Next, we consider how to compute these residues to obtain the homogenous term in a closed form.

Algorithm 2: MellinREtoDE

Input: $\mathcal{M}[I(x);s]$, a recurrence equation $\mathcal{M}[I(x);s]$ satisfies, $\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}s^j u(s+i) = 0$ and the fundamental strip of $I(x)$, $\langle \alpha, \beta \rangle$.

Output: A differential equation with polynomial coefficients which has a homogeneous term that is the sum of residues of the form $\mathrm{Res}(x^{-s+c}s^j\mathcal{M}[I(x);s])$, where c and j are integers, that is satisfied by $I(x)$.

begin
  Set k to be some value in (α, β);
  Set $DE = 0$;
  Set $HTerm = 0$;
  Set $Rec = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}s^j u(s+i)$;
  for *a from n downto 1* **do**
    for *b from m downto 1* **do**
      if *(5.4) does not hold with $c = a$ and $j = b$* **then**
        Algorithm failed;
      end
      Set $DE = DE + c_{a,b}x^a \left(-x\frac{d}{dx}\right)^b I(x)$;
      Set $HTerm = HTerm + c\sum_{k<\Re(s)<k+a} \mathrm{Res}(x^{-s+a}s^b\mathcal{M}[I(x);s])$;
      Set $Rec = Rec - c_{a,b}(s+a)^b u(s+a)$ (Note this changes the values of some of the $c_{i,j}$ in Rec);
    end
  end
  return $DE = HTerm$
end

We can compute the leading terms of the asymptotic expansions of $f(x)$ at zero and $g(x)$ at infinity using the oracle that is assumed to exist. These expansions are used with the theory in Section 4.4 to locate the poles of $\mathcal{M}[I(x);s]$. Now that the poles of $\mathcal{M}[I(x);s]$ are known, sums over regions of values of s can be simplified to a finite number of residue calculations. Often, many of the residues will cancel each other out, so simplifying can be performed prior to any residue calculations being made. If the homogenous term can be simplified to zero, we have a differential equation that is satisfied by $I(x)$ and no additional information is required. Otherwise, we need to calculate the remaining residues. The Mellin transforms of $f(x)$ and $g(x)$ are computed in order to calculate the residues using the other oracle. Note that the theory in Section 4.4 can be used to compute the residues of $I(x)$. Using this theory it is possible, in certain cases, to compute the residues while only computing one of $\mathcal{M}[f(x);s]$ or $\mathcal{M}[g(x);s]$. Once the residues are computed, we obtain a differential equation that $I(x)$ satisfies with an explicit closed form homogeneous term. This term will have the following form:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} d_{i,j} x^{r_i} \left(\log(x)\right)^j$$

where $d_{i,j}$ and r_i are complex numbers.

Since x^{r_i} and $(\log(x))^j$ are holonomic functions themselves, it follows by Theorem 2.2.1 that the homogeneous term is also holonomic. Therefore, we can compute a differential operator to annihilate this term. Applying this operator to both sides of the differential equation that annihilates $I(x)$ yields a linear homogeneous differential equation with polynomial coefficients. This homogeneous differential equation also annihilates $I(x)$.

Example 5.2.1. In Example 5.1.1, we want to compute

$$I(x) = \int_0^\infty K_0(t)\sin(xt)dt$$

with

$$f(x) = K_0(x) \text{ and } g(x) = \sin(x).$$

We had then computed

$$(s+1)w(s+2) + sw(s) = 0$$

as a recurrence equation that annihilates $\mathcal{M}[I(x);s]$.

Applying MellinREtoDE to the recurrence yields

$$(-x^3 - x)I'(x) - x^2 I(x) = \sum_{k<\Re(s)<k+2} \operatorname{Res}(x^{-s+2}(s-1)\mathcal{M}[I(x);s]).$$

Since $k \in (-1,0)$ we are only concerned with poles of $\mathcal{M}[I(x);s]$ located at points

p with $0 < \Re(p) < 2$. Since $g(x) = \sin(x)$ is oscillating at infinity, $\mathcal{M}[g(x); s]$ contributes no poles to $\mathcal{M}[I(x); s]$ in the right half of the complex plane. Therefore we are only required to observe the expansion of $f(x)$ at zero. This is

$$f(x) \sim -\gamma + \ln(2) - \ln(x) + \left(-\frac{1}{4}\gamma + \frac{1}{4}\ln(2) + \frac{1}{4} - \frac{1}{4}\ln(x)\right)x^2 + O(x^3)$$

as $x \to 0$. We see from the expansion that $\mathcal{M}[f(x); 1-s]$ will contribute a double pole to $\mathcal{M}[I(x); s]$ at $s = 1$ and this is the only pole that needs to be considered. Finally, we compute

$$\begin{aligned} \mathcal{M}[f(x); 1-s] &= 2^{-s-1}\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)^2 \text{ and} \\ \mathcal{M}[g(x); s] &= 2^{-1+s}\sqrt{\pi}\frac{\Gamma(\frac{1}{2} + \frac{1}{2}s)}{\Gamma(-\frac{1}{2}s + 1)} \end{aligned}$$

and use these to compute the $\mathrm{Res}_{s=1}(x^{-s+2}(s-1)\mathcal{M}[I(x); s]) = x$. Therefore

$$(-x^3 - x)I'(x) - x^2 I(x) = x$$

is satisfied by $I(x)$. □

5.3 Obtaining Explicit Closed Form Solutions

In previous sections of this chapter, we obtained a method to compute a differential equation that annihilates the integral $I(x)$. This section considers solving the differential equation to obtain an explicit closed form for $I(x)$. However, we do not have any initial conditions to determine which particular solution is equal to $I(x)$. In addition, at 0, where initial conditions are often given, $I(x)$ may not behave nicely. Computing initial conditions at other points also is not feasible. For example, in order to compute initial conditions at $x = 1$, we need to compute

$$\int_0^\infty f(t)g(t)dt,$$

which may not be any easier to compute than $I(x)$.

The first step to solving this problem is to convert the differential equation satisfied by $I(x)$ into a linear homogeneous differential equation with polynomial coefficients. Once we obtain the differential equation, we determine if zero is an ordinary or regular singular point. If zero is an irregular singular point, finding a solution using this point must be abandoned since we do not have the means to compute an asymptotic expansion of $I(x)$ in this case. The distinction between zero being an irregular point or not is similar to deciding which contour is used in the definition of the Meijer G function (Section 3.1). If zero is an ordinary or regular singular point, we can attempt to compute a general solution to the original differential equation. If a general solution cannot be found, another method must be used.

Let $\hat{I}(x)$ be the general solution. By Theorems 4.2.1 and 4.2.2, $\hat{I}(x)$ can be represented as a finite combination of logarithmic sums. Thus, $\hat{I}(x)$ is of the form

$$\hat{I}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{m} c_{i,j} x^{r_i} (\log(x))^j$$

where the $c_{i,j}$ are complex and may depend on the unknown constants, m is a positive integer, and the r_i are complex with $\Re(r_i)$ strictly increasing as i increases. Since $\hat{I}(x)$ is a holonomic function we can use our oracle to compute some leading terms of the above sum. Therefore, we have

$$\hat{I}(x) = \sum_{i=0}^{p} \sum_{j=0}^{m} c_{i,j} x^{r_i} (\log(x))^j + O(x^{r_{p+1}}) \tag{5.5}$$

where we now know the $c_{i,j}$ and the r_i explicitly.

From Section 4.3, the terms in the expansion of $\hat{I}(x)$ correspond to residues of $\mathcal{M}[I(x); s] = \mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]$. This gives

$$\sum_{j=0}^{m} c_{i,j} x^{r_i} (\log(x))^j = \text{Res}_{s=-r_i}(x^{-s}\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]), \tag{5.6}$$

for all i from 1 to p. These equations give linear equations involving different unknown constants from the general solution. Inserting the solved values of the constants into $\hat{I}(x)$ gives an explicit function which is equal to $I(x)$ about zero. The convergence of the solution is only guaranteed along the positive real line from zero to a distance that is equal to the radius of convergence of the right hand side of (5.5). The smallest possible radius of convergence is the distance from zero to the closest singularity of the homogeneous differential equation satisfied by $I(x)$. We consider the convergence of a solution off the positive real line in Chapter 6.

A similar technique is used to compute an explicit solution that is valid about infinity. We first ensure that infinity is not an irregular singular point of the homogeneous differential equation satisfied by $I(x)$. Then, we define $\hat{I}(x)$ exactly the same as above. By similar theory as above, the expansion of $\hat{I}(x)$ at infinity has the form:

$$\hat{I}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{n} d_{i,j} x^{t_i} (\log(x))^j + O(x^{r_{t+1}})$$

where the $d_{i,j}$ are complex and possibly depend on the unknown constants, n is a positive integer, and t_i are complex with $\Re(t_i)$ strictly decreasing. Similarly, we can compute the leading terms of the above expansion and obtain a set of equations of the form:

$$\sum_{j=0}^{n} d_{i,j} x^{-t_i} (\log(x))^j = -\mathrm{Res}_{s=t_i}(x^{-s}\mathcal{M}[f(); 1-s]\mathcal{M}[g(x); s]). \tag{5.7}$$

As before, we can solve for the unknowns in $\hat{I}(x)$ which results in a solution that is valid for large values of x.

Each equation of the form (5.6) or (5.7) may give more than one linear equation involving the unknown coefficients due to $\ln(x)$ appearing. For example, computing the integral

$$\int_0^\infty \ln(1+t^2) Y_0(xt) dt$$

using the methods of this chapter requires one to solve the differential equation

$$x^3 y'''(x) + 4x^2 y''(x) + (-x^3 + 2x) y'(x) - x^2 y(x) = 0. \tag{5.8}$$

The general solution of (5.8) is

$$\begin{aligned} \hat{I}(x) &= \frac{c_1}{x} + c_2 \left(I_0(x) - \frac{\pi}{2}\mathbf{L}_0(x) I_1(x) + \frac{\pi}{2}\mathbf{L}_1(x) I_0(x) \right) \\ &\quad + c_3 \left(K_0(x) + \frac{\pi}{2}\mathbf{L}_1(x) K_0(x) + \frac{\pi}{2}\mathbf{L}_0(x) K_1(x) \right). \end{aligned}$$

and it has a regular singularity at zero and an irregular singularity at infinity. Therefore the closed form solution can be found by observing the initial terms of the expansion of the integral about zero.

$$I(x) \sim \frac{\pi}{x} - 2\ln(2) + 2\gamma - 2 + 2\ln(x) + O(x^2)$$

and

$$\hat{I}(x) \sim \frac{c_1}{x} + c_2 - c_3\gamma + c_3 \ln(2) - c_3 \ln(x) + c_3 + O(x^2).$$

Equating the coeffients of x^{-1} and x^0 gives 3 linear equations,

$$c_1 = \pi, \ \frac{c_1}{x} + c_2 - c_3\gamma - c_3 \ln(x) + c_3 = 2\ln(2) + 2\gamma - 2, \text{ and } -c_3 = 2$$

which implies

$$c_1 = \pi, \ c_2 = 0, \text{ and } c_3 = 0.$$

It is impossible to know a priori how many linear equations each equation of the form (5.6) or (5.7) will yield. Therefore, determining the required minimum number of equations of the form (5.6) or (5.7) needed is difficult. Since we need to determine each coefficient there also must be at least one equation that contains each of the unknowns. Suppose,

$$\hat{I}(x) = y_0(x) + \sum_{i=1}^{n} c_i y_i(x)$$

with

$$y_1(x) = 1 + 2x + O(x^2) \text{ and } y_2(x) = \pi x^{100} + \pi x^{101} + O(x^{102})$$

as $x \to 0$. Computing the leading terms of $\hat{I}(x)$ as $x \to 0$ would yield no linear equations involving c_2 until terms that are $O(x^{100})$ are computed. Let k be the maximum power of x in the leading term of each of the $y_i(x)$s as $x \to 0$. If the first $\Omega(x^{k+\epsilon})$ terms of $\hat{I}(x)$ are computed, this guarantees that every unknown appears in one linear equation. But it does not guarantee that enough linear equations are computed to solve for all the unknowns. Therefore, it is possible that more than the first $\Omega(x^k + \epsilon)$ terms need to be computed. Note, when solving for the unknowns only n linear equations are needed. An equation can be omitted if it is linearly dependent on other equations already obtained.

Example 5.3.1. In Example 5.2.1 we have

$$I(x) = \int_0^\infty K_0(t)\sin(xt)dt$$

and the differential equation for $I(x)$ computed as

$$(-x^3 - x)I'(x) - x^2 I(x) = x. \tag{5.9}$$

Since x is annihilated by the operator $xD_x - 1$, taking that operator and applying it to (5.9) gives

$$(-x^4 - x^2)I''(x) - 3x^3 I'(x) - x^2 I(x) = 0.$$

From the equation above, we observe that i, $-i$ and ∞ are all regular singular points with all other points ordinary. A general solution of (5.9) is

$$\hat{I}(x) = \frac{\operatorname{arcsinh}(x) + a}{\sqrt{x^2+1}}$$

where a is some constant. The leading term of $\hat{I}(x)$ as $x \to 0$ is

$$\hat{I}(x) \sim a + O(x).$$

Therefore we obtain the equation

$$\begin{aligned} a &= \operatorname{Res}_{s=0}(x^{-s}\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]) \\ &= \operatorname{Res}_{s=0}\left(x^{-s}2^{-s-1}\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)^2 2^{-1+s}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(-\frac{1}{2}s+1)}\right) \\ &= 0 \end{aligned}$$

and so

$$I(x) = \frac{\operatorname{arcsinh}(x)}{\sqrt{x^2+1}},$$

valid on $[0, 1)$. The leading term of $\hat{I}(x)$ as $x \to \infty$ is

$$\hat{I}(x) \sim \frac{(\ln(2) + \ln(x) + a)}{x} + O\left(\frac{1}{x^3}\right).$$

Therefore, we have

$$\begin{aligned}
\frac{(\ln(2) + \ln(x) + a)}{x} &= \operatorname{Res}_{s=1}(x^{-s}\mathcal{M}[f(x); 1-s]\mathcal{M}[g(x); s]) \\
&= \operatorname{Res}_{s=1}\left(x^{-s}2^{-s-1}\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)^2 2^{-1+s}\sqrt{\pi}\frac{\Gamma(\frac{1}{2} + \frac{1}{2}s)}{\Gamma(-\frac{1}{2}s + 1)}\right) \\
&= \frac{(\ln(2) + \ln(x))}{x}
\end{aligned}$$

which again implies $a = 0$ and so

$$I(x) = \frac{\operatorname{arcsinh}(x)}{\sqrt{x^2 + 1}}$$

is valid on $[1, \infty)$. Thus,

$$I(x) = \frac{\operatorname{arcsinh}(x)}{\sqrt{x^2 + 1}}$$

for all positive x with the possible exception of the point $x = 1$. $\square$

Chapter 6

Extending the Algorithm Beyond Mellin Transforms

In the previous chapter, a method was devised to compute a differential equation satisfied by the integral

$$I(x) = \int_0^\infty f(t)g(xt)dt.$$

It is also shown that the differential equation can be solved to obtain an explicit function that is equal to the integral. However, there are two major flaws with the method currently given. The first problem is the solution is only obtained for real x. The algorithm could be extended somewhat using just Mellin transforms but the techniques presented in this chapter are easier and more fruitful. The second problem is what to do in cases where the Mellin transform of $f(x)$ or $g(x)$ does not exist. For example consider the following integral,

$$\int_0^\infty I_0(t)K_0(xt)dt$$

which can be shown to be

$$\frac{K\left(\frac{1}{x}\right)}{x}$$

for $\Re(x) > 1$. $I_0(x)$ has no Mellin transform in any normal sense because it is exponentially increasing as $x \to \infty$. The method in the previous chapter would fail when computing such an integral.

To solve these problems, we borrow the idea of using contour integrals as in how the Meijer G is defined in Chapter 3. Suppose a function $g(x)$ can be written as the contour integral

$$g(x) = \frac{1}{2\pi i}\int_C g^*(s)x^{-s}ds$$

where $g^*(s)$ is a meromorphic function defined on the entire complex plane and C is an appropriate path through the complex plane that causes the integral to converge. Instead of using $g(x)$ and its Mellin transform, we use $g(x)$ and $g^*(s)$.

In the cases where the Mellin transform of $g(x)$ exists, $g^*(s)$ may be taken as the Mellin transform. However, $g^*(s)$ is not unique. When the Mellin transform does not exist, the function $g^*(s)$ will have similar properties to a Mellin transform.

All poles of $g^*(s)$ can be classified as one of two types, either a right pole or a left pole. We take the path C to be the path from $-i\infty$ to $i\infty$ such that the path keeps all left poles of $g^*(s)$ on the left of C and all the right poles of $g^*(s)$ on the right of C. Note that all functions $g^*(s)$ must have the same poles, which follows from Cauchy's residue theorem (assuming $g^*(s)$ decreases sufficiently as $|\Im(s)| \to \infty$ for all $\Re(x)$). Thus, we can refer to these poles as the left and right poles of the contour representation of $g(x)$ and not have to specify a particular $g^*(s)$.

Many functions can be written as a contour integral in this fashion. The Meijer G function is a special case of such a contour integral and many special functions can be written as a Meijer G function. Using this idea, we may extend the methods presented in the previous chapter to include many more integrals. The rest of this chapter presents the details on extending the ideas of Chapter 5.

6.1 Annihilating the Integrand of the Contour Integral

In Chapter 5, we developed a method to compute a recurrence equation with polynomial coefficients that annihilates $\mathcal{M}[g(x);s]$, given a linear homogeneous differential equation with polynomial coefficients that is satisfied by $g(x)$. This method is extended to compute a recurrence equation with polynomial coefficients that annihilates $g^*(s)$ given:

$$g(x) = \frac{1}{2\pi i}\int_C g^*(s)x^{-s}ds.$$

Indeed, we can apply the same technique here. This follows as

$$x^a g(x) = \frac{x^a}{2\pi i}\int_C g^*(s)x^{-s}ds = \frac{1}{2\pi i}\int_C g^*(s)x^{-s+a}ds \tag{6.1}$$

and

$$\begin{aligned} g^{(n)}(x) &= \frac{1}{2\pi i}\frac{d^n}{dx^n}\left(\int_C g^*(s)x^{-s}ds\right) && (6.2)\\ &= \frac{1}{2\pi i}\int_C g^*(s)\frac{d^n}{dx^n}\left(x^{-s}\right)ds && (6.3)\\ &= \frac{1}{2\pi i}\int_C (-1)^n\frac{\Gamma(s)}{\Gamma(s-n)}g^*(s)x^{-s+n}ds && (6.4)\end{aligned}$$

which are analogous to the Mellin transform Theorems 2.1.6 and 2.1.7 that were used to justify DEtoMellinRE in the previous chapter. Therefore algorithm DEtoMellinRE computes a recurrence equation that annihilates $g^*(s)$ given a differential equation that is satisfied by $g(x)$.

Noting that

$$\begin{aligned}\frac{1}{x}f\left(\frac{1}{x}\right) &= \frac{1}{2\pi i x}\int_C f^*(s)\left(\frac{1}{x}\right)^{-s} ds \\ &= \frac{1}{2\pi i}\int_C f^*(s)x^{s-1}ds\end{aligned}$$

is analogous to Theorem 2.1.6, computing the differential equation that annihilates $\frac{1}{x}f\left(\frac{1}{x}\right)$ and performing algorithm DEtoMellinRE with it as input results in a recurrence equation that annihilates $f^*(1-s)$.

The next step of the process given in the previous chapter is to compute the Mellin transform of $I(x)$. The generalization of this is to compute $I^*(s)$ such that

$$I(x) = \frac{1}{2\pi i}\int_C I^*(s)x^{-s}ds.$$

Suppose

$$g(x) = \frac{1}{2\pi i}\int_C g^*(s)x^{-s}ds$$

for any valid $g^*(s)$ and

$$f(x) = \frac{1}{2\pi i}\int_C f^*(s)x^{-s}ds$$

where $f^*(s)$ will be defined below. In order to compute $I^*(s)$, we update Parseval's Formula, Theorem 2.1.1. Suppose C is a path which divides all the left poles of $f^*(1-s)$ and $g^*(s)$ from the right poles of $f^*(1-s)$ and $g^*(s)$. Then,

$$\begin{aligned}\int_0^\infty f(t)g(xt)dt &= \int_0^\infty f(t)\frac{1}{2\pi i}\left(\int_C g^*(s)x^{-s}t^{-s}ds\right)dt \\ &= \frac{1}{2\pi i}\int_C\left(\int_0^1 f(t)t^{-s}dt\right)g^*(s)x^{-s}ds \qquad (6.5)\\ &+ \frac{1}{2\pi i}\int_C\left(\int_1^\infty f(t)t^{-s}dt\right)g^*(s)x^{-s}ds. \qquad (6.6)\end{aligned}$$

Suppose that the change of integration is valid. If $f(x)$ does not increase exponentially as $x \to 0$, (6.5) can be considered a Mellin transform (with respect to $1-s$ not s) of

$$f_1(x) = \begin{cases} f(x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise}\end{cases}.$$

Therefore, the first integral will converge absolutely for $s < c_1$ where c_1 is determined by the behaviour of $f(x)$ as $x \to 0$. Similarly, for (6.6), if $f(x)$ does not increase exponentially as $x \to \infty$ the integral can be seen as a Mellin transform of

$$f_2(x) = \begin{cases} f(x) & \text{if } 1 < x \\ 0 & \text{otherwise} \end{cases},$$

and so the second integral will converge absolutely for $s > c_2$ where c_2 is determined by the behaviour of $f(x)$ as $x \to \infty$. Then, Mellin transforms of $f_1(x)$ and $f_2(x)$ can each be continued into the entire complex plane using the theory in Section 4.3. We define

$$f^*(1-s) = \mathcal{M}[f_1(x); 1-s] + \mathcal{M}[f_2(x); 1-s]$$

where $\mathcal{M}[f_1(x); s]$ and $\mathcal{M}[f_2(x); s]$ are the analytically continued versions of the Mellin transform.

The inverse Mellin transform of $\mathcal{M}[f_1(x); 1-s] + \mathcal{M}[f_2(x); 1-s]$ may not exist in the normal sense. If $c_1 > c_2$, the Mellin transform of $f(x)$ will exist in the normal sense and we can use the normal inverse Mellin transform to separate the left and right poles of $f^*(s)$. If $c_1 \leq c_2$, we can perform the inverse Mellin transforms

$$\frac{1}{2\pi i} \int_{-i\infty+k_1}^{i\infty+k_1} \mathcal{M}[f_1(x); 1-s] ds$$

and

$$\frac{1}{2\pi i} \int_{-i\infty+k_2}^{i\infty+k_2} \mathcal{M}[f_2(x); 1-s] ds$$

where $k_1 < c_1$ and $c_2 < k_2$. All poles of $\mathcal{M}[f_1(x); 1-s]$ will be to the right of k_1, and all poles of $\mathcal{M}[f_2(x); 1-s]$ will be to the left of k_2, which are the right and left poles of $\mathcal{M}[f(x); 1-s]$, respectively. Thus, we have defined the left and right poles of $f^*(1-s)$. Note, if the set of poles for $\mathcal{M}[f_1(x); 1-s]$ and those of $\mathcal{M}[f_2(x); 1-s]$ intersect, then $f^*(1-s)$ cannot be defined in this manner.

If we can justify the change of integration made above, then we obtain the generalized Parseval Formula,

$$\int_0^\infty f(t) g(xt) dt = \frac{1}{2\pi i} \int_C f^*(1-s) g^*(s) x^{-s} ds, \tag{6.7}$$

with the restriction that $f(x)$ has a generalized Mellin transform. This is true assuming $f(x)$ does not grow exponentially as $x \to 0$ or as $x \to \infty$ and the left and right poles of $f(x)$'s generalized Mellin transform do not coincide. The contour C starts at $-i\infty$ and goes to $i\infty$ and separates the left and right poles of $f^*(1-s)$ as defined above and also at the same time separates the left and right poles of $g^*(s)$. The generalized Parseval Formula is only valid when both the left and right hand side integrals converge.

Since the Mellin transforms of $f_1(x)$ and $f_2(x)$ converge absolutely, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \left(\int_0^1 \left|f(t)t^{-s}\right|dt\right)\left|g^*(s)x^{-s}\right|ds &\quad + \\
\frac{1}{2\pi i}\int_C \left(\int_1^\infty \left|f(t)t^{-s}\right|dt\right)\left|g^*(s)x^{-s}\right|ds &\quad = \\
\frac{1}{2\pi i}\int_C \mathcal{M}[|f(t)|\,;1-s]\,\left|g^*(s)x^{-s}\right|ds &\quad =
\end{aligned}$$

$$\frac{1}{2\pi i}\int_C \mathcal{M}[|f(t)|\,;1-s]\,|g^*(s)|\,|x|^{-\Re(s)}\,e^{\arg(x)\Im(s)}ds.$$

Therefore, if σ is the $\Re(s)$ as $\Im(s)\to\infty$, τ along C is the $\Re(s)$ as $\Im(s)\to-\infty$ along C,

$$\begin{aligned}
f^*(1-\sigma+iy) &\sim O\left(e^{-\theta_p y}\right), \\
f^*(1-\tau-iy) &\sim O\left(e^{\theta_n y}\right), \\
g^*(\sigma+iy) &\sim O\left(e^{-\phi_p y}\right), \text{ and} \\
g^*(\tau-iy) &\sim O\left(e^{\phi_n y}\right)
\end{aligned} \tag{6.8}$$

as $y\to\infty$, the integral will converge for

$$-(\theta_n+\phi_n) < \arg(x) < (\theta_p+\phi_p). \tag{6.9}$$

It may be possible to include equality in (6.9) by looking at more detailed expansions than those given in (6.8).

The generalized Parseval Formula can be justified by first expanding $g(xt)$ as a contour integral in (6.5) and (6.6). In this case, the formula will hold if $g(x)$ does not grow exponentially as $x\to 0$ or as $x\to\infty$ and the left poles of $\mathcal{M}[g_1(x);s]$ and right poles of $\mathcal{M}[g_2(x);s]$ do not coincide, where $g_1(x)$ and $g_2(x)$ are defined similarly to $f_1(x)$ and $f_2(x)$. A contour C that divides the left and right poles of both $f^*(1-s)$ and $g^*(s)$ must also exist. The integral $I(x)$ does not exist if both $f(x)$ and $g(x)$ grow exponentially as $x\to 0$ or as $x\to\infty$, so this generalization covers many possible cases. With the generalized Parseval Formula, it is now justified to compute the recurrence equation that annihilates $I^*(s)$ in the same manner as computing the recurrence equation that annihilated the Mellin transform of $I(x)$ in the previous chapter.

6.2 Generalizing MellinREtoDE

The algorithm MellinREtoDE presented in Chapter 5 recovers the differential equation that the integral $I(x)$ satisfies given the recurrence equation that annihilates the Mellin transform of $I(x)$ and the ability to compute residues of $\mathcal{M}[I(x);s]$. Similarly, a method is needed to recover the differential equation that the integral $I(x)$ satisfies given the recurrence relation that annihilates $I^*(s)$. First we generalize Theorem 5.2.1 to the following theorem.

Theorem 6.2.1. *Let $I(x)$ be a function with*

$$I(x) = \frac{1}{2\pi i} \int_C I^*(s) x^{-s} ds$$

where C is defined in the beginning of this chapter and also, let

$$\lim_{\Im(s)\to\infty} s^j I^*(s) = 0 \text{ for all } s \text{ such that } \Re(s) \in [\sigma, \sigma + c] \tag{6.10}$$

and

$$\lim_{\Im(s)\to-\infty} s^j I^*(s) = 0 \text{ for all } s \text{ such that } \Re(s) \in [\tau, \tau + c] \tag{6.11}$$

where $\sigma = \Re(s)$ as $\Im(s) \to \infty$ along C and $\tau = \Re(s)$ as $\Im(s) \to -\infty$ along C. Then,

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= \frac{1}{2\pi i} \int_C x^{-s}(s+c)^j I^*(s) ds \\ &- \sum_{s\in R} Res\left(x^{-s+c} s^j I^*(s)\right), \end{aligned}$$

where R is the set of poles between the contours C and $C + c$ where $C + c$ is the contour obtained from C by shifting it a distance c to the right.

Proof. By (6.1) and (6.2),

$$\left(-x\frac{d}{dx}\right)^j I(x) = \frac{1}{2\pi i} \int_C s^j I^*(s) x^{-s} ds.$$

Therefore,

$$x^c \left(-x\frac{d}{dx}\right)^j I(x) = x^c \int_C s^j I^*(s) x^{-s} ds = \int_C s^j I^*(s) x^{-s+c} ds.$$

By Cauchy's Residue Theorem and the assumed behaviour of $I^*(s)$ as $|\Im(s)| \to \infty$,

$$\begin{aligned} x^c \left(-x\frac{d}{dx}\right)^j I(x) &= \int_{C+c} s^j I^*(s) x^{-s+c} ds \\ &\quad - \sum_{s\in R} \text{Res}\left(x^{-s+c} s^j I^*(s)\right). \end{aligned}$$

Finally, substituting $s' = s - c$ into the above integral gives

$$x^c \left(-x\frac{d}{dx}\right)^j I(x) = \int_C (s'+c)^j I^*(s'+c)x^{-s'}ds' - \sum_{s\in R} \text{Res}\left(x^{-s+c}s^j I^*(s)\right).$$

□

Using Theorem 6.2.1, we obtain a generalized MellinREtoDE algorithm. The proof of correctness is similar to the MellinREtoDE algorithm in Chapter 5, replacing the use of Theorem 5.2.1 with that of Theorem 6.2.1.

Algorithm 3: Generalized MellinREtoDE

Input: $I^*(s)$ and a recurrence equation $I^*(s)$ satisfies,
$\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}s^j u(s+i) = 0$

Output: A differential equation with polynomial coefficients with a homogeneous term that is the sum of a few residues of the form $\text{Res}(x^{-s+c}s^j I^*(s))$, where c and j are integers, that is satisfied by $I(x)$.

```
begin
    Set DE = 0;
    Set HTerm = 0;
    Set Rec = Σ_{i=0}^{n} Σ_{j=0}^{m} c_{i,j} s^j u(s+i);
    for a from n downto 1 do
        for b from m downto 1 do
            if (6.10) or (6.11) does not hold with c = a and j = b then
                Algorithm failed;
            end
            Set DE = DE + c_{a,b} x^a (-x d/dx)^b I(x);
            Set HTerm =
            HTerm + c (Σ_{s∈R} Res(x^{-s+a} s^b I*(s) - Σ_{s∈A} Res(x^{-s+a} s^b I*(s)))
            (R is the set of poles encircling C and not C + a, and A is the set
            of poles encircling C + a and not C);
            Set Rec = Rec - c_{a,b} (s+a)^b u(s+a) (Note this changes some of
            the c_{i,j}'s values in Rec);
        end
    end
    return DE = HTerm
end
```

Example 6.2.1. Consider computing the differential equation that annihilates

$$I(x) = \int_0^\infty e^t K_m(xt)dt.$$

Let

$$\begin{aligned} f(x) &= e^x \text{ and} \\ g(x) &= K_m(x). \end{aligned}$$

Note that $\frac{1}{x} f\left(\frac{1}{x}\right)$ satisfies the differential equation

$$x^2 y'(x) + (x+1)y(x) = 0$$

while $g(x)$ satisfies the differential equation

$$x^2 y''(x) + xy'(x) + (-x^2 - m^2)y(x) = 0.$$

Performing the DEtoMellinRE algorithm on the two differential equations above outputs

$$u(s) - su(s+1) = 0$$

and

$$-u(s+2) + (s^2 - m^2)u(s) = 0$$

respectively. Thus, the recurrence equation that annihilates $I^*(x)$ is

$$(-s^2 + m^2)u(s) + (s^2 + s)u(s+2) = 0. \tag{6.12}$$

Performing the generalized MellinREtoDE algorithm on (6.12) gives

$$(-x^2 + x^4)y''(x) + (4x^3 - x)y'(x) + (m^2 + 2x^2)y(x) = 0.$$

Thus, we obtain a differential equation satisfied by $I(x)$.

6.3 Obtaining Explicit Solutions Using the Contour Integral

Finally we want to obtain an explicit closed form of $I(x)$ by solving the differential equation that is obtained by the techniques of the previous sections of this chapter. The only difference between the method covered in Chapter 5 and the generalized case is to replace $\mathcal{M}[I(x); s]$ with $I^*(s)$. The theory in Section 4.3 that gives a method of computing expansions of functions in terms of the residues of Mellin transforms easily extends to the general case. The residues of the left and right poles of $I^*(s)x^{-s}$ will correspond to the leading terms of the expansion of $I(x)$ as $x \to 0$ and $x \to \infty$, respectively. We can observe this by taking the generalized Parseval Formula (6.7) and shifting the contour. Note that these expansions will only be valid where the integral with the shifted contour exists.

However, a few small problems arise. Where is the calculated solution valid and where is the integral $I(x)$ valid? In Chapter 5, we are guaranteed $I(x)$ exists for

$x > 0$ and the calculated solution also exists for $x > 0$. This guarantee no longer holds and we also wish to compute solutions that are valid in regions of the complex plane.

The calculation of $I(x)$ is valid where the generalized Parseval formula (6.7) (and possibly a shifted version of it) is valid, and also in regions where the differential equation obtained has formal solutions that converge. Determining the regions of validity for the integral $I(x)$ is preformed by observing the behavior of the integral $I(x)$ as $t \to 0, \infty$. Any point that is a singularity of the integral on $(0, \infty)$ will give the requirements for the integral to exist. For example, in Example 6.2.1, the integrand is

$$e^t K_m(xt).$$

The leading behaviour of the integrand as $t \to \infty$ is dominated by the factor

$$\frac{1}{\sqrt{t}} \left(\frac{1}{e^t}\right)^{\Re(x)-1}.$$

Therefore, one requirement for the integral to exist is $\Re(x) > 1$. The leading behaviour of the integrand as $t \to 0$ is dominated by $t^{-|m|}$ if $m \neq 0$ and $\ln(t)$ if $m = 0$. Another requirement for the integral to exist is therefore $|m| < 1$. There are no other singularities of the integrand and so these are the only requirements for the integral to exist. Therefore, the integral exists if

$$\Re(x) > 1 \text{ and } |\Re(m)| < 1. \tag{6.13}$$

There is no guarantee that the calculated solution will exist everywhere the integral $I(x)$ actually converges. Also it is possible that the calculated solution will exist where the integral $I(x)$ does not exist. This is caused by using analytic continuations thoughout the calculations. Using a different $f^*(s)$ or $g^*(s)$ it is possible to obtain differential equations that are valid in different regions of the complex plane. Using these different equations it is possible to compute a more complete closed form solution.

Let's now complete the computation that we started in Example 6.2.1.

Example 6.3.1. Consider computing the differential equation that annihilates

$$I(x) = \int_0^\infty e^t K_m(xt)dt.$$

We therefore have

$$f(x) = e^x \text{ and } g(x) = K_m(x).$$

In Example 6.2.1 we calculated

$$(-x^2 + x^4)y''(x) + (4x^3 - x)y'(x) + (m^2 + 2x^2)y(x) = 0 \tag{6.14}$$

as a differential equation satisfied by $I(x)$. Let

$$f_1^*(s) = \Gamma(s)(-1)^{-s},\ f_2^*(s) = \Gamma(s)\cos(\pi s),\ f_3^*(s) = \Gamma(s)(-1)^s,$$

and

$$g^*(s) = 2^{s-2}\Gamma(\frac{1}{2}s + \frac{1}{2}m)\Gamma(\frac{1}{2}s - \frac{1}{2}m).$$

Note that using any of $f_1^*(s), f_2^*(s)$ or $f_3^*(s)$ in Example 6.2.1 results in (6.14) being the result. Then,

$$\begin{aligned}
f_1^*(x+yi) &\sim (-1+i)\sqrt{\pi}y^{(-i)y}e^{\frac{3}{2}i\pi x+iy-\frac{3}{2}\pi y}\left(\frac{1}{y}\right)^{x-\frac{1}{2}} \\
f_1^*(x-yi) &\sim (-1-i)\sqrt{\pi}\,(-y)^{(-i)y}\,e^{\frac{1}{2}ix\pi+iy-\frac{1}{2}\pi y}\left(-\frac{1}{y}\right)^{x-\frac{1}{2}} \\
f_2^*(x+yi) &\sim \left(\frac{1}{2}-\frac{1}{2}i\right)\sqrt{\pi}y^{iy}e^{\left(-\frac{1}{2}i\right)x\pi+(-i)y+\frac{1}{2}\pi y}\left(\frac{1}{y}\right)^{\frac{1}{2}-x} \\
f_2^*(x-yi) &\sim \left(\frac{1}{2}+\frac{1}{2}i\right)\sqrt{\pi}\,(-y)^{iy}\,e^{\frac{1}{2}ix\pi+(-i)y-\frac{1}{2}\pi y}\left(-\frac{1}{y}\right)^{\frac{1}{2}-x} \\
f_3^*(x+yi) &\sim (-1+i)\sqrt{\pi}y^{(-i)y}e^{\left(-\frac{1}{2}i\right)x\pi+iy+\frac{1}{2}\pi y}\left(\frac{1}{y}\right)^{x-\frac{1}{2}} \\
f_3^*(x-yi) &\sim (-1-i)\sqrt{\pi}\,(-y)^{(-i)y}\,e^{\left(-\frac{3}{2}i\right)\pi x+iy+\frac{3}{2}\pi y}\left(-\frac{1}{y}\right)^{x-\frac{1}{2}} \\
g^*(x+yi) &\sim (-i)\,y^{iy-1}e^{\frac{1}{2}ix\pi+(-i)y-\frac{1}{2}\pi y}\pi\left(\frac{1}{y}\right)^{-x} \\
g^*(x-yi) &\sim i\,(-y)^{iy-1}\,e^{\left(-\frac{1}{2}i\right)x\pi+(-i)y+\frac{1}{2}\pi y}\pi\left(-\frac{1}{y}\right)^{-x}
\end{aligned}$$

as $y \to \infty$. Define

$$\begin{aligned}
I_1(x) &= \int_C f_1^*(1-s)g^*(s)ds, \\
I_2(x) &= \int_C f_2^*(1-s)g^*(s)ds \text{ and} \\
I_3(x) &= \int_C f_3^*(1-s)g^*(s)ds.
\end{aligned}$$

By the expansions given above, $I_1(x)$ exists for $-2\pi < \arg(x) < 0$ and $I_3(x)$ exists for $0 < \arg(x) < 2\pi$. $I_2(x)$ will exist for $\arg(x) = 0$ as a principal value integral.

Solving (6.14) gives

$$\frac{c_1\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{\frac{1}{2}m}+c_2\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{-\frac{1}{2}m}}{\sqrt{-1+x^2}} \tag{6.15}$$

as a global solution. Equation (6.14) only has regular singularities -1, 0, 1 and ∞. By (6.13) the integral only exists for $\Re(x) > 1$. Therefore, we only need to consider the formal solution of (6.14) at infinity as it will converge for $|x| > 1$. Note that all integrands of $I_1(x)$, $I_2(x)$ and $I_3(x)$ have identical right poles. This is because the poles of each $f_i^*(1-s)g^*(s)$ on the right are at positive integers and the only difference between them are the $(-1)^{-s}$, $\cos(\pi s)$ and $(-1)^s$ factors which are equivalent at positive integers. Hence, for $\Re(x) > 1$, $I_1(x) = I_2(x) = I_3(x)$ where they exist. The leading terms of (6.15) as $x \to \infty$ are

$$\frac{c_1+c_2}{x}+\frac{-imc_1+imc_2}{x^2}+O\left(\frac{1}{x^3}\right).$$

This must equal

$$\operatorname{Res}_{s=1}(I_1^*(s)x^{-s})+\operatorname{Res}_{s=2}(I_1^*(s)x^{-s}) = \\ -\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}+\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m\right)}{x}+\frac{\frac{1}{4}\Gamma\left(\frac{1}{2}m\right)m^2\Gamma\left(-\frac{1}{2}m\right)}{x^2}.$$

Equating these and solving for c_1 and c_2 gives

$$-\frac{\frac{1}{4}\pi}{\sqrt{-1+x^2}\sin\left(\frac{1}{2}\pi m\right)\cos\left(\frac{1}{2}\pi m\right)}\left(-\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{\frac{1}{2}m}\sin\left(\frac{1}{2}\pi m\right)+\right. \\ (-i)\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{\frac{1}{2}m}\cos\left(\frac{1}{2}\pi m\right)-\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{-\frac{1}{2}m}\sin\left(\frac{1}{2}\pi m\right) \\ \left.+i\left(\frac{\sqrt{-1+x^2}-i}{\sqrt{-1+x^2}+i}\right)^{-\frac{1}{2}m}\cos\left(\frac{1}{2}\pi m\right)\right) \tag{6.16}$$

after simplifying. Using (6.13) we have (6.16) as the answer for $\Re(x) > 1$ and $|\Re(m)| < 1$.

6.4 A Method to Compute Integrals

Below is an outline of the steps required to compute an integral using the techniques of this chapter.

1. Compute and list the singularities of the integrand $f(t)g(xt)$ on $t \in (0, \infty)$.

Include zero and infinity. These are the points to consider the behaviour of $f(t)g(xt)$ to determine if $I(x)$ exists.

2. For each singularity, compute the leading term of the asymptotic expansion of $f(t)g(xt)$ as t approaches that singularity.

3. From the expansions, determine if $I(x)$ exists for some x and what conditions must be made to ensure its existance.

4. Determine the functions $f^*(s)$ such that

$$f(x) = \int_C f^*(s)x^{-s}ds,$$

where C is the contour discussed in this chapter. These contour integrals influence where the calculations below are valid. Each function $f^*(s)$ should result in solutions that are valid in different sectors of the complex plane or there is no need to include them.

5. Compute the leading terms of the expansion of each function $f^*(x+yi)$ as $y \to \infty$ and then as $y \to -\infty$.

6. From the expansions in Step 5, determine the sectors where each contour integral

$$f(x) = \int_C f^*(s)x^{-s}ds$$

is valid.

7. In a similar manner, determine the functions $g^*(s)$ such that

$$g(x) = \int_C g^*(s)x^{-s}ds.$$

Again, multiple functions might be needed for similar reasons explained in Step 4.

8. Compute the leading terms of the expansion of each function $g^*(x+yi)$ as $y \to \infty$ and then as $y \to -\infty$.

9. From the expansions, determine the sectors where each contour integral

$$g(x) = \int_C g^*(s)x^{-s}ds.$$

is valid.

10. Using the $f^*(s)$s and $g^*(s)$s, compute one or more $I(x)$s such that

$$I(x) = \int_0^\infty f(t)g(xt)dt = \int_C I^*(s)x^{-s}ds$$

by the extended Parseval's formula. List where each $I(x)$ is valid.

11. Compute a differential equation that is satisfied by $\frac{1}{x}f\left(\frac{1}{x}\right)$.
12. Compute a differential equation that is satisfied by $g(x)$.
13. Compute a recurrence equation that annihilates all $f^*(1-s)$ by using the DEtoMellinRE algorithm. The input is the differential equation that $\frac{1}{x}f\left(\frac{1}{x}\right)$ satisfies.
14. Compute a recurrence equation that annihilates all $g^*(s)$ by using the DEtoMellinRE algorithm. For input, use the differential equation that $g(x)$ satisfies.
15. Compute a recurrence equation that annihilates all $I^*(s) = f^*(1-s)g^*(s)$ using the term-by-term product closure property of holonomic sequences.
16. Compute the differential equation satisfied by $I(x)$ using the MellinREtoDE algorithm on the recurrence equation that $I^*(s)$ satisfies. Note that the algorithm has to compute residues of $I^*(s)$ so this calculation should be performed for each $I^*(s)$.
17. For each differential equation that is satisfied by $I(x)$, if the equation has a non-zero source term, compute a differential operator that annihilates this term. Apply that operator on the right hand side of rest of the differential equation. This results in a homogeneous differential equation that is satisfied by $I(x)$.
18. For each homogeneous equation satisfied by $I(x)$, compute its regular and irregular singularities.
19. Compute a global solution for each differential equation that $I(x)$ satisfies (using the original differential equation).
20. For each differential equation $I(x)$ satisfies, if zero is not an irregular singularity of the homogeneous differential equation, then compute the leading terms of the asymptotic expansion of the corresponding global solution at zero.
21. For each expansion, sum for each power p of x in the expansion

    $$\mathrm{Res}_{s=-p}\left(I^*(s)x^{-s}\right)$$

    using the corresponding $I^*(s)$. Note that if $-p$ is not on the left side of the contour, the residue calculation is replaced with zero.
22. Equating the calculations from the previous two steps forms a system of linear equations that determines the unknown constants for each global solution. Determine the constants for each global solution.

23. Determine the region of validity of the solution by computing the region from zero to the next closest singularity of the differential equation. The solution will be valid in the disk where both $I^*(s)$ and the integral $I(x)$ are valid.

24. For each differential equation that $I(x)$ satisfies, if infinity is not an irregular singularity of the homogeneous differential equation, then compute the leading terms of asymptotic expansion of the corresponding global solution at infinity.

25. For each expansion, sum for each $-p$ power of x in the expansion

$$\mathrm{Res}_{s=p}\left(I^*(s)x^{-s}\right)$$

using the corresponding $I^*(s)$. Note that if p is not on the right side of the contour, the residue calculation is replaced with 0.

26. Equating the calculations from the previous two steps forms a system of linear equations that determines the unknown constants in the corresponding differential equation's global solution. Compute these constants.

27. Determine this solution's region of validity by computing the region from infinity to the next closest singularity of the differential equation. The solution will be valid in the region where both $I^*(s)$ and the integral $I(x)$ are valid.

28. Combine all the information from the previous steps to obtain a closed form solution of the integral.

Chapter 7

Conclusion

In this thesis we present a method for the computation of integrals with the form of

$$\int_0^\infty f(t)g(xt)dt,$$

as first presented by Bruno Salvy in 2000 in his talk, "A Symbolic Algorithm for the Computation of Convolution Integrals", [16]. We present two methods for doing this with one being an extension of the other. Both methods compute a differential equation that is satisfied by the integral.

The first method uses the theory of Mellin transforms to compute the differential equation. How to compute the explicit solution of the integral by solving the differential equation is also presented. While this method succeeds at computing many different integrals it suffers from several problems. The most obvious problem is if one of the functions $f(x)$ or $g(x)$ does not have a Mellin transform then the method fails. There are many integrals that involve functions that do not have Mellin transforms so this is a very limiting assumption. Another assumption that is made is the fundamental strips of the Mellin transforms of $f(x)$ and $g(x)$ must overlap. Finally the computations only consider the solution as a real value of x.

The second method in some sense uses a generalization of Mellin transforms. This extended method overcomes the problems of the first. More functions turn out to have one or more of these generalized Mellin transforms. The theory of the generalized Mellin transforms also removes the idea of a critical strip so there is no need for the critical strip of $f(x)$ and $g(x)$ to overlap. Finally the theory allows us to compute the solution for complex values of x.

One of the major problems with the methods presented in this thesis is the fact that they are not complete algorithms. We use the term method thoughout the thesis and what we really mean by method is a series of steps done by a human with the aid of a computer. There are several steps presented in Section 6.4 that are non-trivial to implement as an algorithm. In many of the steps one must determine the exponential or algebraic rate of decay/increase of a function. While there exist methods for computing the leading terms of a function as a variable approaches a limit point it is still difficult to determine how the function behaves algorithmically. While this is not a research problem because other algorithms such as the ones that compute limits already do similar calculations, it would still be an interesting endeavour to automate these steps.

The other problem is given a function, say $f(x)$, compute a function $f^*(s)$ such that

$$f(x) = \int_C f^*(s)x^{-s}ds.$$

There are techniques to calculate such an $f^*(s)$ such as computing the Mellin transform of $f(x)$ or expressing $f(x)$ and a Meijer G function and taking the Gamma functions that are the integrand of the definition of the Meijer G function. However sometimes more than one $f^*(s)$ is needed to obtain the entire solution to the integral problem; See Appendix B for some examples. Each of the above techniques to compute $f^*(s)$ will give the same $f^*(s)$. For the examples in Appendix B when multiple $f^*(s)$ were needed human observation was used to find them by modifying the $f^*(s)$ obtained by the two techniques previously mentioned.

One final observation that has not been explored is that the differential equation that is obtained by the methods of this thesis could be solved numerically. Solving the differential equation numerically would be a way to solve the integral numerically. Some integrals are difficult to numerically integrate while linear ordinary differential equations are easy to solve numerically and the solutions are obtained very fast. There could potentially exist many problems with this idea. However it definitely is an avenue for exploration.

APPENDICES

Appendix A

Computable Integral Transforms

The following are some of the most well known and important integral transforms that can be computed by the methods presented in this Thesis.

Transform	$\int_0^\infty f(t)g(xt)dt$
Fourier	$\int_{-\infty}^\infty f(t)\frac{1}{\sqrt{2\pi}}e^{-ixt}dt = \int_0^\infty f(t)\frac{1}{\sqrt{2\pi}}e^{-ixt}dt + \int_0^\infty f(-t)\frac{1}{\sqrt{2\pi}}e^{ixt}dt$
Laplace	$\int_0^\infty f(t)e^{-xt}dt$
Two-sided Laplace	$\int_{-\infty}^\infty f(t)e^{-xt}dt = \int_0^\infty f(t)e^{-xt}dt + \int_0^\infty f(-t)e^{xt}dt$
Fourier Cosine	$\int_0^\infty f(t)\sqrt{\frac{2}{\pi}}\cos(xt)dt$
Fourier Sine	$\int_0^\infty f(t)\sqrt{\frac{2}{\pi}}\sin(xt)dt$
Hankel	$\int_0^\infty f(t)t\mathrm{J}_v(xt)dt$
Hartley	$\int_0^\infty f(t)\frac{1}{\sqrt{2\pi}}\left(\sin(xt)+\cos(xt)\right)dt$

Appendix B

Example Calculations

In this chapter we will present a series of example calculations. These examples were chosen to demonstrate a wide range of the different types of calculations that may be needed to be made when applying the theory of the previous chapters. The examples will all follow the outline of the calculations given in section 6.4.

B.1 Example 1: $\int_0^\infty \frac{\cos(xt)}{1+t^2} dt$

For this example we will consider the problem of computing the integral

$$I(x) = \int_0^\infty \frac{\cos(xt)}{1+t^2} dt.$$

Therefore we have

$$f(x) = \frac{1}{1+x^2} \text{ and } g(x) = \cos(x).$$

1. (a) 0
   (b) ∞
2. (a) 1
   (b) $\frac{e^{t(i\Re(x)+|\Im(x)|)}}{2t^2}$
3. $\Im(x) = 0$
4. $\frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}$
5. $i\pi e^{\frac{\pi}{2}(ix-i-y)}$ as $y \to \infty$ and $-i\pi e^{-\frac{\pi}{2}(ix-i-y)}$ as $y \to -\infty$
6. $-\frac{\pi}{2} \le \arg(x) \le \frac{\pi}{2}$
7. $\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$

8. $\frac{1}{2}\sqrt{2}e^{-\frac{1}{4}i(-4\ln(y)\Im(x)+\pi-4y\ln(y)+4y)}\sqrt{\pi}\left(\frac{1}{y}\right)^{1/2-\Re(x)}$ as $y \to \infty$ and $\frac{1}{2}\sqrt{2}e^{\frac{1}{4}i(4\ln(-y)\Im(x)+\pi+4y\ln(y)-4y)}\sqrt{\pi}\left(-\frac{1}{y}\right)^{1/2-\Re(x)}$ as $y \to -\infty$

9. $\arg(x) = 0$

10. $\frac{\pi\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{2\sin\left(\frac{\pi(1-s)}{2}\right)}$ with $-\frac{\pi}{2} \le \arg(x) \le \frac{\pi}{2}$

11. $(x + x^3)y'(x) - (1 + x^2)y(x) = 0$

12. $y''(x) + y(x) = 0$

13. $(-1 - s)u(s) + (-1 - s)u(s + 2) = 0$

14. $u(s + 2) + (s^2 + s)u(s) = 0$

15. $(-s^2 - s)u(s) + u(s + 2) = 0$

16. $x^2y(x) - x^2y''(x) = 0$

17. $x^2y(x) - x^2y''(x) = 0$

18. ∞ is a irregular singularity

19. $c_1e^x + c_2e^{-x}$

20. $c_1 + c_2 + (c_1 - c_2)x + O(x^2)$

21. $\frac{\pi}{2} - \frac{\pi}{2}x$

22. $c_1 = 0, c_2 = \frac{\pi}{2}$

23. $x \ge 0$

24. Omit

25. Omit

26. Omit

27. Omit

28. $I(x) = \frac{\pi}{2}e^{-x}$ for $x \ge 0$

Note that this calculation only computes the solution for $x \ge 0$ but $I(x)$ exists for all x such that $\Im(x) = 0$. We can obtain the solution for the negative x by noticing that $\cos(x)$ is an odd function therefore

$$I(x) = \frac{\pi}{2}e^{-|x|}$$

for $\Im(x) = 0$

B.2 Example 2: $\int_0^\infty tY_0(t)K_0(t)J_1(tx)I_1(tx)dt$

Consider computing the integral

$$I(x) = \int_0^\infty tY_0(t)K_0(t)J_1(tx)I_1(tx)dt.$$

Therefore we have

$$f(x) = xY_0(x)K_0(x) \text{ and } g(x) = J_1(x)I_1(x).$$

1. (a) 0
   (b) ∞
2. (a) $-\frac{x^2 \ln\left(\frac{1}{t}\right)^2 t^3}{2\pi}$
   (b) $\left(-\frac{1}{8} - \frac{1}{8}i\right)\left(e^{-t}\right)^{-\Re(x)-|\Im(x)|+1}$ (large factor that doesn't matter)
3. $\Im(x) = 0$ and $\Re(x) < 1$
4. $-\frac{\left(\frac{\sqrt(2)}{4}\right)^{(}-s)\sqrt(2)\Gamma\left(\frac{1}{4}+\frac{s}{4}\right)^3}{8\sqrt(\pi)\Gamma\left(\frac{1}{4}-\frac{s}{4}\right)}$
5. $-\frac{2^{-\frac{1}{2}+\left(-\frac{1}{2}i\right)y+\frac{3}{2}x}\sqrt{\pi}\sqrt{1-i}y^{iy}e^{\left(-\frac{1}{4}y(4i+\pi)\right)}\left(\frac{1}{4}i\right)^{\frac{3}{4}x}\left(\frac{1}{y}\right)^{\frac{1}{2}-x}(-1/4i)^{\frac{1}{4}x}}{\sqrt{-1+i}}$ as $y \to \infty$
   and $-\frac{2^{-\frac{1}{2}+\left(-\frac{1}{2}i\right)y+\frac{3}{2}x}\sqrt{\pi}\sqrt{1-i}(-y)^{iy}e^{\left(-\frac{1}{4}y(4i-\pi)\right)}\left(-\frac{1}{4}i\right)^{\frac{3}{4}x}\left(-\frac{1}{y}\right)^{\frac{1}{2}-x}(1/4i)^{\frac{1}{4}x}}{\sqrt{-1-i}}$ as $y \to -\infty$
6. $-\frac{\pi}{4} \leq \arg(x) \leq \frac{\pi}{4}$
7. $\frac{\sqrt{\pi}\Gamma\left(\frac{s}{4}+\frac{1}{2}\right)\left(\frac{\sqrt{2}}{4}\right)^{-s}}{4\Gamma\left(1-\frac{s}{4}\right)\Gamma\left(\frac{1}{2}-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}-\frac{s}{4}\right)}$
8. $\frac{\left(-\frac{1}{2}+\frac{1}{2}i\right)y^{i(\Im(x)+y)}2^{\left(-\frac{1}{2}i\right)y+\frac{1}{2}}e^{-\frac{1}{4}y(4i-\pi)}\left(\frac{1}{y}\right)^{-\Re(x)+\frac{3}{2}}\left(-\frac{1}{4}i\right)^{\frac{3}{4}x}16i^{\frac{1}{4}x}}{\sqrt{\pi}}$ as $y \to \infty$ and
   $\frac{\left(-\frac{1}{2}-\frac{1}{2}i\right)(-y)^{i(\Im(x)+y)}2^{\left(-\frac{1}{2}i\right)y+\frac{1}{2}}e^{-\frac{1}{4}y(4i+\pi)}\left(-\frac{1}{y}\right)^{-\Re(x)+\frac{3}{2}}\frac{1}{4}i^{\frac{3}{4}x}(-16i)^{\frac{1}{4}x}}{\sqrt{\pi}}$
9. $-\frac{\pi}{4} \leq \arg(x) \leq \frac{\pi}{4}$
10. $-\frac{\frac{1}{32}\left(\frac{1}{4}\sqrt{2}\right)^{-1+s}\sqrt{2}\Gamma\left(\frac{1}{2}-\frac{1}{4}s\right)^2\Gamma\left(\frac{1}{4}s+\frac{1}{2}\right)\left(\frac{1}{4}\sqrt{2}\right)^{-s}}{\Gamma\left(\frac{1}{4}s\right)\Gamma\left(1-\frac{1}{4}s\right)\Gamma\left(\frac{3}{2}-\frac{1}{4}s\right)}$ where $-\frac{\pi}{2} \leq \arg(x) \leq \frac{\pi}{2}$
11. $x^8y^{(4)}(x) + 16x^7y'''(x) + 73x^6y''(x) + 103x^5y'(x) + (32x^4+4)y(x) = 0$
12. $x^3y^{(4)}(x) + 4x^2y'''(x) - 3xy''(x) + 3y'(x) + 4x^3y(x) = 0$
13. $4u(s) + (s^4 + 6s^3 + 12s^2 + 8s)u(s+4) = 0$

14. $4u(s+4)+(s^4+2s^3-4s^2-8s)u(s)=0$

15. $(-s+2)u(s)+(s+2)u(s+4)=0$

16. $(-x^5+x)y'(x)-(2x^4+2)y(x)=\frac{2x^2}{\pi}$

17. $(-x^6+x^2)y''(x)+(-5x^5+x)y'(x)-(4x^4+4)y(x)=0$

18. $-1,0,1,\infty,I,-I$ are regular singularities

19. $\frac{c_1}{x^2}+\frac{-\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x+1)-\frac{1}{2}\ln\left(x^2+1\right)}{\pi x^2}$

20. $\frac{c_1-\frac{1}{2}i}{x^2}+\frac{\frac{1}{2}x^2}{\pi}+O\left(x^4\right)$

21. $\frac{x^2}{2\pi}$

22. $c_1=\frac{i}{2}$

23. $\Im(x)=0$ and $\Re(x)<1$

24. Omit

25. Omit

26. Omit

27. Omit

28. $I(x)=\frac{\frac{1}{2}i}{x^2}+\frac{-\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x+1)-\frac{1}{2}\ln\left(x^2+1\right)}{\pi x^2}$ for $\Im(x)=0$ and $\Re(x)<1$

Note that we Omit the calculations we did because the solution at infinity doesn't need to be computed since the solution at 0 gives us the complete solution.

B.3 Example 3: $\int_0^\infty \frac{tJ_m(xt)^2}{(a^2-t^2)(b^2-t^2)}dt$

In this example we will compute the integral

$$I(x)=\int_0^\infty \frac{tJ_m(xt)^2}{(a^2-t^2)(b^2-t^2)}dt.$$

Therefore we have

$$f(x)=\frac{x}{(a^2-x^2)(b^2-x^2)} \text{ and } g(x)=J_m(x)^2.$$

1. (a) 0

   (b) ∞

(c) $\pm a$ if $a \in \mathbb{R}$

(d) $\pm b$ if $b \in \mathbb{R}$

2. (a) $\frac{x^{m\,2} t^{2m+1}}{\Gamma(m+1)^2 e^{\ln(2) m\,2} a^2 b^2}$

(b) $\frac{\frac{1}{2} i e^{i\pi m + 2\Im(x)t + (-2i)t\Re(x)}}{\pi x t^4}$ if $\Im(x) \geq 0$ and $\frac{\left(-\frac{1}{2}i\right) e^{(-i)\pi m + 2it\Re(x) - 2\Im(x)t}}{\pi x t^4}$ if $\Im(x) < 0$

(c) $-\frac{\frac{1}{2} J_m(\pm xa)^2}{(a^2-b^2)(a\mp t)}$

(d) $-\frac{\frac{1}{2} J_m(\pm xb)^2}{(a^2-b^2)(b\mp t)}$

If $a = b$ then the leading term is $\frac{\frac{1}{4} J_m(\pm xa)^2}{a(a\mp t)^2}$ which is another case.

3. $m > \frac{3}{2}, \Im(x) = 0$ and $a, b \notin \mathbb{R}$
If $a, b \notin \mathbb{R}$ then the integral will exist as a principal value integral assuming $m > \frac{3}{2}, \Im(x) = 0$ and $a \neq b$. $\frac{\frac{1}{2}\pi \cot\left(\frac{1}{2}\pi(1+s)\right)\left(b^{2\,\frac{1}{2}s-\frac{1}{2}} - a^{2\,\frac{1}{2}s-\frac{1}{2}}\right)}{a^2-b^2}$
If $a = \pm b$ then one must use $\frac{\frac{1}{2}\left(-a^2\right)^{\frac{1}{2}s+\frac{1}{2}} B\left(\frac{1}{2}s+\frac{1}{2}, \frac{3}{2}-\frac{1}{2}s\right)}{a^4}$.

4. $\frac{\left(-\frac{1}{2}i\int_0^\infty \frac{tJ_m(xt)^2}{(a^2-t^2)(b^2-t^2)}dt\right)\pi e^{(-i)y\ln(|a|)} e^{-\frac{1}{2}x\ln\left(a^2\right)}\left(\frac{1}{e^{\frac{1}{2}\pi y}}\right)^{-\frac{\arg\left(a^2\right)}{\pi}}}{a^2-b^2}$ as $y \to \infty$ and $\frac{\left(-\frac{1}{2}i\right)\pi e^{(-i)y\ln(|b|)} e^{-\frac{1}{2}x\ln\left(b^2\right)}\left(\frac{1}{e^{\frac{1}{2}\pi y}}\right)^{-\frac{\arg\left(b^2\right)}{\pi}}}{a^2-b^2}$ as $y \to -\infty$ which is true if $\arg(a^2) > \arg(b^2)$ otherwise the roles of a and b must be reversed. If $a = \pm b$ then $-\frac{|a|^{iy}\left(-a^2\right)^{\frac{1}{2}x+\frac{1}{2}}\frac{1}{2}i^{1+\frac{1}{2}x}\left(-\frac{1}{2}i\right)^{-\frac{1}{2}x}\pi y\left(e^{-\frac{1}{2}\pi y}\right)^{\frac{\pi+\arg\left(-a^2\right)}{\pi}}}{a^4}$ as $y \to \infty$ and $\frac{\left(-\frac{1}{2}i\right)^{1+\frac{1}{2}x}|a|^{iy}\left(-a^2\right)^{\frac{1}{2}x+\frac{1}{2}}\frac{1}{2}i^{-\frac{1}{2}x}\pi y\left(e^{\frac{1}{2}\pi y}\right)^{\frac{\pi-\arg\left(-a^2\right)}{\pi}}}{a^4}$

5. $\frac{1}{2}\arg(b^2) \leq \arg(x) \leq \frac{1}{2}\arg(a^2)$
If $a = b$ then $-\frac{\pi}{2} - \frac{1}{2}\arg(-a^2) \leq \arg(x) \leq \frac{\pi}{2} + \frac{1}{2}\arg(-a^2)$ $\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma\left(m+\frac{1}{2}s\right)}{\sqrt{\pi}\Gamma\left(-\frac{1}{2}s+1\right)\Gamma\left(m-\frac{1}{2}s+1\right)}$

6. $\frac{(1+i)\left(-\frac{1}{2}i\right)^{-m+\frac{1}{2}x} y^{\frac{1}{2}i(\Im(2m+x)+2y+\Im(-2m+x))}\left(\frac{1}{2}i\right)^{m+\frac{1}{2}x} 2^{(-i)y} e^{(-i)y}\left(\frac{1}{y}\right)^{\frac{3}{2}-\frac{1}{2}\Re(2m+x)-\frac{1}{2}\Re(-2m+x)}}{\sqrt{\pi}}$ as $y \to \infty$ and
$\frac{(1-i)\left(\frac{1}{2}i\right)^{-m+\frac{1}{2}x}(-y)^{\frac{1}{2}i(\Im(2m+x)+2y+\Im(-2m+x))}\left(-\frac{1}{2}i\right)^{m+\frac{1}{2}x} 2^{(-i)y} e^{(-i)y}\left(-\frac{1}{y}\right)^{\frac{3}{2}-\frac{1}{2}\Re(2m+x)-\frac{1}{2}\Re(-2m+x)}}{\sqrt{\pi}}$
as $y \to -\infty$

7. $\arg(x) = 0$

8. $\frac{\frac{1}{4}\sqrt{\pi}\int_0^\infty \frac{tJ_m(xt)^2}{(a^2-t^2)(b^2-t^2)}dt \cot\left(\frac{1}{2}\pi(2-s)\right)\left(\left(b^2\right)^{-\frac{1}{2}s} - \left(a^2\right)^{-\frac{1}{2}s}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma\left(m+\frac{1}{2}s\right)}{(a^2-b^2)\Gamma\left(-\frac{1}{2}s+1\right)\Gamma\left(m-\frac{1}{2}s+1\right)}$ where $\frac{1}{2}\arg(b^2) \leq \arg(x) \leq \frac{1}{2}\arg(a^2)$

If $a = b$ then $\frac{\frac{1}{4}\left(-a^2\right)^{\frac{1}{2}s+\frac{1}{2}} B\left(\frac{1}{2}s+\frac{1}{2},\frac{3}{2}-\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma\left(m+\frac{1}{2}s\right)}{a^4\sqrt{\pi}\Gamma\left(-\frac{1}{2}s+1\right)\Gamma\left(m-\frac{1}{2}s+1\right)}$ where $-\frac{\pi}{2} - \frac{1}{2}\arg(-a^2) \leq \arg(x) \leq \frac{\pi}{2} + \frac{1}{2}\arg(-a^2)$.

9. $(a^2b^2x^5 - x^3a^2 - x^3b^2 + x)y'(x) + 2a^2b^2x^4 - 2 = 0$

10. $x^2y'''(x) + 3xy''(x) + (4x^2 + 1 - 4m^2)y'(x) + 4xy(x) = 0$

11. $(-a^2b^2s - 2a^2b^2)u(s+4) + ((a^2+b^2)s + 2a^2 + 2b^2)u(s+2) + (-2-s)u(s) = 0$

12. $(-4 - 4s)u(s+2) + (-s^3 + 4m^2s)u(s) = 0$

13. $(48a^2b^2 + 64a^2b^2s + 16s^2a^2b^2)u(s+4) + ((4b^2 + 4a^2)s^4 + (28a^2 + 28b^2)s^3 + (-16m^2b^2 + 72b^2 + 72a^2 - 16m^2a^2)s^2 + (80b^2 - 48m^2a^2 + 80a^2 - 48m^2b^2)s - 32m^2a^2 + 32b^2 + 32a^2 - 32m^2b^2)u(s+2) + (s^6 + 6s^5 + (-8m^2 + 12)s^4 + (-32m^2 + 8)s^3 + (-48m^2 + 16m^4)s^2 + (32m^4 - 32m^2)s)u(s)$

14. $x^6y^{(6)} + 9x^5y^{(5)} + ((4b^2 + 4a^2)x^6 + (-8m^2 + 17)x^4)y^{(4)} + ((28a^2 + 28b^2)x^5 + (4 - 16m^2)x^3)y''') + (16a^2b^2x^6 + (40a^2 + 40b^2 - 16m^2b^2 - 16m^2a^2)x^4 + (-8m^2 + 16m^4 + 1)x^2)y^{(\prime\prime)} + (80a^2b^2x^5 + (-32m^2a^2 - 32m^2b^2 + 8a^2 + 8b^2)x^3 + (8m^2 - 16m^4 - 1)x)y'(x) + (48a^2b^2x^4)y(x) = 0$

15. $x^6y^{(6)} + 9x^5y^{(5)} + ((4b^2 + 4a^2)x^6 + (-8m^2 + 17)x^4)y^{(4)} + ((28a^2 + 28b^2)x^5 + (4 - 16m^2)x^3)y''') + (16a^2b^2x^6 + (40a^2 + 40b^2 - 16m^2b^2 - 16m^2a^2)x^4 + (-8m^2 + 16m^4 + 1)x^2)y^{(\prime\prime)} + (80a^2b^2x^5 + (-32m^2a^2 - 32m^2b^2 + 8a^2 + 8b^2)x^3 + (8m^2 - 16m^4 - 1)x)y'(x) + (48a^2b^2x^4)y(x) = 0$

16. 0 is a regular singularity and ∞ is an irregular singularity

We omit the rest of the calculations because the differential equation obtained is very complicated and computing a global solution seems out of reach. However a solution of the integral is known to be $\frac{-J_m(xb)Y_m(xb)+J_m(ax)Y_m(ax))}{b^2-a^2}$ which satisfies the differential equation calculated.

B.4 Example 4: $\int_0^\infty \frac{\sqrt{t}\sin(xt)}{1+t}dt$

Consider the integral

$$I(x) = \int_0^\infty \frac{\sqrt{t}\sin(xt)}{1+t}dt.$$

Therefore we have

$$f(x) = \frac{\sqrt{x}}{1+x} \text{ and } g(x) = \sin(x).$$

1. (a) 0
   (b) ∞

2. (a) $xt^{\frac{3}{2}}$

   (b) $\frac{\frac{1}{2}ie^{-t(i\Re(x)-\Im(x))}}{\sqrt{t}}$ if $\Im(x) \geq 0$ and $\frac{\left(-\frac{1}{2}i\right)e^{t(i\Re(x)-\Im(x))}}{\sqrt{t}}$ if $\Im(x) < 0$

3. $\Im(x) = 0$

4. $\pi \csc\left(\frac{1}{2}\pi\left(1+2s\right)\right)$

5. $2i\pi e^{\frac{1}{2}\pi(2ix-3i-2y)}$ as $y \to \infty$ and $(-2i)\,\pi e^{-\frac{1}{2}\pi(2ix-3i-2y)}$ as $y \to -\infty$

6. $-\pi < \arg(x) < \pi$

7. $\Gamma(s)\sin\left(\frac{1}{2}\pi s\right)$

8. $\frac{1}{2}i\sqrt{2}\sqrt{\pi}e^{\left(-\frac{1}{4}i\right)(\pi-4y\ln(y)+4y)}\left(\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y \to \infty$ and $\left(-\frac{1}{2}i\right)\sqrt{2}\sqrt{\pi}e^{\frac{1}{4}i(\pi+4y\ln(-y)-4y)}\left(-\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y \to -\infty$

9. $\arg(x) = 0$

10. $\pi\csc\left(\frac{1}{2}\pi\left(3-2s\right)\right)\Gamma\left(s\right)\sin\left(\frac{1}{2}\pi s\right)$ where $-\pi < \arg(x) < \pi$

11. $(2x+2x^2)y'(x) + (3x+1)y(x) = 0\ln(x)J_0(x$

12. $y''(x) + y(x) = 0$

13. $(1-2s)u(s) + (1-2s)u(s+1)$

14. $u(s+2) + (s^2+s)u(s) = 0$

15. $u(s+2) + (s^2+s)u(s) = 0$

16. $x^2y(x) + x^2y''(x) = \frac{\frac{1}{8}\sqrt{2}\sqrt{\pi}(3+2x)}{\sqrt{x}}$

17. $(6x^3+4x^4)y'''(x)+(6x^3+15x^2)y''(x)+(6x^3+4x^4)y'(x)+(6x^3+15x^2)y(x) = 0$

18. $0, -3/2$ are regular singularities and ∞ is an irregular singularity

19. $\sin\left(x\right)c_2{+}\cos\left(x\right)c_1 1{+}\frac{-\pi\sqrt{x}(\sin(x)+\cos(x))C\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)-\pi\sqrt{x}(\sin(x)-\cos(x))S\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)+\frac{1}{2}\sqrt{2}\sqrt{\pi}}{\sqrt{x}}$

20. $\frac{\frac{1}{2}\sqrt{2}\sqrt{\pi}}{\sqrt{x}} + c_1 - \sqrt{x}\sqrt{\pi}\sqrt{2} + c_2x + O\left(x^{\frac{3}{2}}\right)$

21. $\frac{\frac{1}{2}\sqrt{2}\sqrt{\pi}}{\sqrt{x}} - \sqrt{x}\sqrt{\pi}\sqrt{2} + \pi x$

22. $c_1 = 0, C_2 = \pi$

23. By looking at the singularities of the differential equation we find that the solution exists for at least $0 \leq x < \frac{3}{2}$ but if one computes the formal solution at 0 one would see all sums are alternating and decreasing thus the solution exists for $x \geq 0$.

24. Omit

25. Omit

26. Omit

27. Omit

28. $\sin\left(x\right)\pi + \frac{-\pi\sqrt{x}(\sin(x)+\cos(x))C\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)-\pi\sqrt{x}(\sin(x)-\cos(x))S\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)+\frac{1}{2}\sqrt{2}\sqrt{\pi}}{\sqrt{x}}$ for $x \geq 0$

The integral is actually valid for all real x but we can use the fact that sin is an odd function to obtain the full solution
$\operatorname{sgn}(x)\left(\sin\left(x\right)\pi + \frac{-\pi\sqrt{x}(\sin(x)+\cos(x))C\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)-\pi\sqrt{x}(\sin(x)-\cos(x))S\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)+\frac{1}{2}\sqrt{2}\sqrt{\pi}}{\sqrt{x}}\right)$ where $x \in \mathbb{R}$.

B.5 Example 5: $\int_0^\infty \sin(t)^2 \ln(xt) J_0(xt)dt$

For this example we will consider computing the integral

$$I(x) = \int_0^\infty \sin(t)^2 \ln(xt) J_0(xt)dt.$$

There we have

$$f(x) = \sin(x)^2 \text{ and } g(x) = \ln(x)J_0(x).$$

1. (a) 0
   (b) ∞

2. (a) $-ln\left(\frac{1}{t}\right)t^2$
   (b) $\frac{\frac{1}{2}+\frac{1}{2}i\sin(t)^2 e^{-t(i\Re(x)-\Im(x))}\ln(t)}{\sqrt{\pi}\sqrt{x}\sqrt{t}}$ if $\Im(x) \geq 0$ and
   $\frac{\frac{1}{2}-\frac{1}{2}i\sin(t)^2 e^{t(i\Re(x)-\Im(x))}\ln(t)}{\sqrt{\pi}\sqrt{x}\sqrt{t}}$ if $\Im(x) < 0$

3. $\Im(x) = 0$

4. $-\frac{\frac{1}{2}\sqrt{\pi}\Gamma\left(1+\frac{1}{2}s\right)}{s\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}$

5. $\left(-\frac{1}{8}-\frac{1}{8}i\right)\sqrt{\pi}\left(-\frac{1}{2}i\right)^{-\frac{1}{2}x} e^{iy(-\ln(y)+\ln(2)+1)}\left(\frac{1}{y}\right)^{x-\frac{1}{2}}\left(\frac{1}{2}i\right)^{-\frac{1}{2}x}$ as $y \to \infty$ and
   $\left(-\frac{1}{8}+\frac{1}{8}i\right)\sqrt{\pi}\frac{1}{2}i^{-\frac{1}{2}x}e^{(-i)y(\ln(-y)-\ln(2)-1)}\left(-\frac{1}{y}\right)^{x-\frac{1}{2}}\left(-\frac{1}{2}i\right)^{-\frac{1}{2}x}$ as $y \to -\infty$

6. $\arg(x) = 0$

7. $\frac{\frac{1}{4}\left(\frac{1}{2}\right)^{-s}\Gamma\left(\frac{1}{2}s\right)\left(2\ln(2)+\Psi\left(\frac{1}{2}s\right)+\Psi\left(-\frac{1}{2}s+1\right)\right)}{\Gamma\left(-\frac{1}{2}s+1\right)}$

8. $\frac{2^{x+iy}\frac{1}{2}i^{\frac{1}{2}x}e^{iy(\ln(y)-\ln(2)-1)}\ln(y)\frac{1}{y}^{-x}\left(-\frac{1}{2}i\right)^{\frac{1}{2}x}}{y}$ as $\to\infty$ and
$-\frac{2^{x+iy}\left(-\frac{1}{2}i\right)^{\frac{1}{2}x}e^{(-i)y(-\ln(-y)+\ln(2)+1)}\ln(-y)\left(-\frac{1}{y}\right)^{-x}\left(\frac{1}{2}i\right)^{\frac{1}{2}x}}{y}$ as $y\to-\infty$

9. $\arg(x)=0$

10. $-\frac{\frac{1}{8}\sqrt{\pi}\Gamma\left(\frac{3}{2}-\frac{1}{2}s\right)\left(\frac{1}{2}\right)^{-s}\left(2\ln(2)+\Psi\left(\frac{1}{2}s\right)+\Psi\left(-\frac{1}{2}s+1\right)\right)}{(1-s)\Gamma\left(-\frac{1}{2}s+1\right)}$ where $\arg(x)=0$

11. $x^5y'''(x)+9x^4y''(x)+(4x+18x^3)y'(x)+(6x^2+4)y(x)=0$

12. $x^3y^{(4)}(x)+4x^2y'''(x)+(2x^3+x)y''(x)+(4x^2-1)y'(x)+x^3y(x)=0$

13. $(-s^3+s)u(s+2)+(4-4s)u(s)=0$

14. $u(s+4)+(4+2s^2+6s)u(s+2)+(s^4+2s^3)u(s)=0$

15. $16s^2u(s)+(-8s^2-16s-8)u(s+2)+(s^2+4s+3)u(s+4)=0$

16. $(x^6+16x^2-8x^4)y''(x)+(5x^5+16x-24x^3)y'(x)+(3x^4-8x^2)y(x)=0$

17. $(x^6+16x^2-8x^4)y''(x)+(5x^5+16x-24x^3)y'(x)+(3x^4-8x^2)y(x)=0$

18. $-2,0,2,\infty$ are regular singularities

19. $\frac{c_1}{\sqrt{x^2-4}}+\frac{c_2(\ln(x+2)-2\ln(x)+\ln(x-2))}{\sqrt{x^2-4}}$

20. $\left(-\frac{1}{2}i\right)c_1+(-i)\,c_2\ln(2)+(-i)\,c_2\ln\left(\frac{1}{x}\right)+\frac{1}{2}c_2\pi+O(x)$

21. $\frac{\pi}{8}$

22. $c_1=\frac{i\pi}{4}, c_2=0$

23. $0\le x<2$

24. $\frac{c_1}{x}+\frac{2c_1-4c_2}{x^3}+O\left(x^{-4}\right)$

25. $\frac{-\frac{1}{2}\ln(2)-\frac{1}{2}\gamma}{x}+\frac{2-\gamma-\ln(2)}{x^3}$

26. $c_1=\frac{1}{2}\left(\ln(2)+\gamma\right), c_2=\frac{1}{2}$

27. $x>2$

28. $I(x)=\begin{cases}\frac{\frac{1}{4}i\pi}{\sqrt{x^2-4}} & \text{if } 0\le x<2\\ \frac{\frac{1}{2}\ln(2)+\frac{1}{2}\gamma}{\sqrt{x^2-4}}+\frac{\frac{1}{2}\ln(x+2)-\ln(x)+\frac{1}{2}\ln(x-2)}{\sqrt{x^2-4}} & \text{if } x>2\end{cases}$

We're missing the calculations for $x < 2$. In this case $g^*(s)$ would need to be replaced with $\frac{\frac{1}{4}\Gamma\left(\frac{1}{2}s\right)^2\left(2^{1+s}\ln(2)\sin\left(\frac{1}{2}\pi s\right)+2^{1+s}\Psi\left(\frac{1}{2}s\right)\sin\left(\frac{1}{2}\pi s\right)+\pi 2^s\cos\left(\frac{1}{2}\pi s\right)+2i\pi 2^s\sin\left(\frac{1}{2}\pi s\right)\right)}{\pi}$. The only other calculations that would change in this case using $g^*(s)$ are the calculations for the initial conditions. The solution for negative x is

$$I(x) = \begin{cases} \frac{-\frac{1}{2}\ln(2)-\frac{1}{2}\gamma+\frac{1}{2}i\pi}{x} & \text{if } -2 < x \leq 0 \\ \frac{-\frac{1}{2}\ln(2)-\frac{1}{2}\gamma+\frac{1}{2}i\pi}{\sqrt{x^2-4}} + \frac{-\frac{1}{2}\ln(x+2)+\ln(x)-\frac{1}{2}\ln(x-2)}{\sqrt{x^2-4}} & \text{if } x < -2 \end{cases}$$

This gives the complete solution

$$I(x) = \begin{cases} \frac{\frac{1}{2}\ln(2)+\frac{1}{2}\gamma}{\sqrt{x^2-4}} + \frac{\frac{1}{2}\ln(x+2)-\ln(x)+\frac{1}{2}\ln(x-2)}{\sqrt{x^2-4}} & \text{if } x > 2 \\ \frac{\frac{1}{4}i\pi}{\sqrt{x^2-4}} & \text{if } 0 \leq x < 2 \\ \frac{-\frac{1}{2}\ln(2)-\frac{1}{2}\gamma+\frac{1}{2}i\pi}{x} & \text{if } -2 < x <= 0 \\ \frac{-\frac{1}{2}\ln(2)-\frac{1}{2}\gamma+\frac{1}{2}i\pi}{\sqrt{x^2-4}} + \frac{-\frac{1}{2}\ln(x+2)+\ln(x)-\frac{1}{2}\ln(x-2)}{\sqrt{x^2-4}} & \text{if } x < -2 \end{cases}.$$

B.6 Example 6: $\int_0^\infty \ln(t)e^{-t}e^{-tx}dt$

Consider the integral

$$I(x) = \int_0^\infty \ln(t)e^{-t}e^{-tx}dt.$$

Therefore we have

$$f(x) = \ln(x)e^{-x} \text{ and } g(x) = e^{-x}.$$

1. (a) 0
   (b) ∞
2. (a) $\ln(t)$
   (b) $\ln(t)\, e^{(-i)t\Im(x)} \left(\frac{1}{e^t}\right)^{\Re(x)+1}$
3. $\Re(x) > -1$
4. $\Psi(s)\Gamma(s)$
5. $1 - i\sqrt{\pi}y^{(-i)y}e^{\frac{1}{2}i\pi x+iy-\frac{1}{2}\pi y}\ln(y)\left(\frac{1}{y}\right)^{x-\frac{1}{2}}$ as $y \to \infty$ and
   $1 + i\sqrt{\pi}\,(-y)^{(-i)y}\, e^{\left(-\frac{1}{2}i\right)\pi x+iy+\frac{1}{2}\pi y}\ln(-y)\left(-\frac{1}{y}\right)^{x-\frac{1}{2}}$ as $y \to -\infty$
6. $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$
7. $\Gamma(s)$

8. $(1-i)\sqrt{\pi}y^{iy}e^{\frac{1}{2}i\pi x+(-i)y-\frac{1}{2}\pi y}\left(\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y\to\infty$ and
$(1+i)\,(-y)^{iy}\,e^{(-i)y+\left(-\frac{1}{2}i\right)\pi x+\frac{1}{2}\pi y}\sqrt{\pi}\left(-\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y\to-\infty$

9. $-\frac{\pi}{2}<\arg(x)<\frac{\pi}{2}$

10. $\Psi(1-s)\Gamma(1-s)\Gamma(s)$ where $-\pi<\arg(x)<\pi$

11. $x^4y''(x)+(3x^3-2x^2)y'(x)+(-x+1+x^2)y(x)=0$

12. $y'(x)+y(x)=0$

13. $u(s)+(1+s^2+2s)u(s+2)+(1+2s)u(s+1)=0$

14. $u(s+1)-su(s)=0$

15. $su(s)+(1+2s)u(s+1)+(s+1)u(s+2)=0$

16. $(-x^3-2x^2-x)y'(x)-(x^2+x)y(x)=x$

17. $(-x^4-2x^3-x^2)y''(x)+(-3x^2-3x^3)y'(x)-x^2y(x)=0$

18. $-1,\infty$ are regular singular points

19. $\frac{-\ln(x+1)+c_1}{x+1}$

20. $c_1+O\left(x\right)$

21. $-\gamma$

22. $c_1=-\gamma$

23. $-\pi<\arg(x)<\pi, |x|<1$ and $\Re(x)>-1$

24. $\frac{-\ln(x)+c_1}{x}+O\left(x^{-2}\right)$

25. $\frac{\gamma+\ln(x)}{x}$

26. $c_1=-\gamma$

27. $-\pi<\arg(x)<\pi$ and $|x|>1$

28. $I(x)=\frac{-\ln(x+1)-\gamma}{x+1}$ for $-\pi<\arg(x)<\pi$ and $\Re(x)>-1$. Note there could possibly be a problem with x such that $|x|=1$ because we have not considered checking if the solutions we calculated exist for these points. This type of calculation can be difficult. For this question the solutions do actually equal the integral at all points x such that $|x|=1$ and $-\pi<\arg(x)<\pi$.

We have not computed the integral for $x<0$ in this case and it will not exist for $x=-1$ but through a series of tedious calculations similar to those above it is possible to show that the integral is also equal to $\frac{-\ln(x+1)-\gamma}{x+1}$ for $-1<x\leq 0$. Thus the solution is $I(x)=\frac{-\ln(x+1)-\gamma}{x+1}$ for $\Re(x)>-1$.

B.7 Example 7: $\int_0^\infty \frac{J_0(t)}{1-xt}dt$

In this example we will compute the integral

$$I(x) = \int_0^\infty \frac{J_0(t)}{1 - xt}dt.$$

Therefore we have

$$f(x) = J_0(x) \text{ and } g(x) = \frac{1}{1 - x}.$$

1. (a) 0
   (b) ∞
   (c) $\frac{1}{x}$ if $x > 0$
2. (a) 1
   (b) $\frac{\left(\frac{\frac{1}{2}\sqrt{2}\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)e^{(-i)t}}{\sqrt{\pi}}+\frac{\frac{1}{2}\sqrt{2}\left(\frac{1}{2}\sqrt{2}+\left(-\frac{1}{2}i\right)\sqrt{2}\right)e^{it}}{\sqrt{\pi}}\right)\left(\frac{1}{t}\right)^{\frac{3}{2}}}{x}$
   (c) $\frac{J_0\left(\frac{1}{x}\right)}{1-xt}$
3. $\arg(x) \neq 0$, however if the integral is taken as a principal value integral it will exist for all x.
4. $\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{-s}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{2}s+1\right)}$
5. $2^{(-i)y-x}\left(-\frac{1}{2}i\right)^{-\frac{1}{2}x}e^{iy(-\ln(y)+\ln(2)+1)}\left(\frac{1}{y}\right)^x\left(\frac{1}{2}i\right)^{-\frac{1}{2}x}$ as $y \to \infty$ and
   $2^{(-i)y-x}\left(\frac{1}{2}i\right)^{-\frac{1}{2}x}e^{iy(-\ln(-y)+\ln(2)+1)}\left(-\frac{1}{y}\right)^x\left(-\frac{1}{2}i\right)^{-\frac{1}{2}x}$ as $y \to -\infty$
6. $\arg(x) = 0$
7. (a) $\pi\cot(\pi s) + i\pi$
   (b) $\pi\cot(\pi s)$
   (c) $\pi\cot(\pi s) - i\pi$
8. (a) $\frac{(-2i)e^{ix\pi}\pi}{e^{(-i)x\pi}(e^{\pi y})^2}$ as $y \to \infty$ and $2i\pi$ as $y \to -\infty$
   (b) $-i\pi$ as $y \to \infty$ and $i\pi$ as $y \to -\infty$
   (c) $-2i\pi$ as $y \to \infty$ and $\frac{2ie^{(-i)x\pi}\pi}{e^{ix\pi}(e^{-\pi y})^2}$ as $y \to -\infty$
9. (a) $-2\pi < \arg(x) < 0$
   (b) $\arg(x) = 0$
   (c) $0 < \arg(x) < 2\pi$

10. (a) $(\pi\cot(\pi s) + i\pi)\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{-s}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{2}s+1\right)}$ where $-2\pi < \arg(x) < 0$

    (b) $\pi\cot(\pi s)\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{-s}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{2}s+1\right)}$ where $\arg(x) = 0$

    (c) $(\pi\cot(\pi s) - i\pi)\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{-s}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{2}s+1\right)}$ where $0 < \arg(x) < 2\pi$

11. $x^4y''(x) + 3x^3y'(x) + (x^2+1)y(x) = 0$

12. $(-1+x)y'(x) + y(x) = 0$

13. $u(s) + (1+s^2+2s)u(s+2) = 0$

14. $-su(s+1) + su(s) = 0$

15. $u(s) + (1+s^2+2s)u(s+2) = 0$

16. $x^4y''(x) + 3x^3y'(x) + (x^2+1)y(x) = 1$, note we get the same differential equation no matter which $I^*(s)$ used.

17. $x^4y'''(x) + 7x^3y''(x) + (10x^2+1)y'(x) + 2xy(x) = 0$

18. ∞ is a regular singularity and 0 is an irregular singularity

19. $\frac{J_0\left(\frac{1}{x}\right)c_2}{x} + \frac{Y_0\left(\frac{1}{x}\right)c_1}{x} + \frac{1}{2}\frac{\pi\mathbf{H}_0\left(\frac{1}{x}\right)}{x}$

20. Omit

21. Omit

22. Omit

23. Omit

24. $\frac{c_2\pi + 2c_1\gamma - 2c_1\ln(2) - 2c_1\ln(x)}{\pi x} + O\left(x^{-2}\right)$

25. (a) $\frac{\gamma - \ln(2) - \ln(x)}{x}$

    (b) $\frac{\gamma - \ln(2) - \ln(x)}{x}$

    (c) $\frac{\gamma - \ln(2) - \ln(x)}{x}$

26. (a) $c_1 = \frac{\pi}{2}, c_2 = i\pi$

    (b) $c_1 = \frac{\pi}{2}, c_2 = 0$

    (c) $c_1 = \frac{\pi}{2}, c_2 = -i\pi$

27. (a) $0 < \arg(x) < 2\pi$

    (b) Is not valid unless the integral is considered as a principal value integral and then it is valid if $\arg(x) = 0$

(c) $-2\pi < \arg(x) < 0$

28.

$$I(x) = \begin{cases} \frac{J_0\left(\frac{1}{x}\right)(-i\pi)}{x} + \frac{Y_0\left(\frac{1}{x}\right)\frac{\pi}{2}}{x} + \frac{1}{2}\frac{\pi \mathbf{H}_0\left(\frac{1}{x}\right)}{x} & \text{if } 0 < \arg(x) \\ \frac{J_0\left(\frac{1}{x}\right)i\pi}{x} + \frac{Y_0\left(\frac{1}{x}\right)\frac{\pi}{2}}{x} + \frac{1}{2}\frac{\pi \mathbf{H}_0\left(\frac{1}{x}\right)}{x} & \text{if } \arg(x) < 0 \end{cases}$$

and if the integral is taken a principal value integral then

$$I(x) = \begin{cases} \frac{J_0\left(\frac{1}{x}\right)(-i\pi)}{x} + \frac{Y_0\left(\frac{1}{x}\right)\frac{\pi}{2}}{x} + \frac{1}{2}\frac{\pi \mathbf{H}_0\left(\frac{1}{x}\right)}{x} & \text{if } 0 < \arg(x) \\ \frac{Y_0\left(\frac{1}{x}\right)\frac{\pi}{2}}{x} + \frac{1}{2}\frac{\pi \mathbf{H}_0\left(\frac{1}{x}\right)}{x} & \text{if } \arg(x) = 0 \\ \frac{J_0\left(\frac{1}{x}\right)i\pi}{x} + \frac{Y_0\left(\frac{1}{x}\right)\frac{\pi}{2}}{x} + \frac{1}{2}\frac{\pi \mathbf{H}_0\left(\frac{1}{x}\right)}{x} & \text{if } \arg(x) < 0 \end{cases} \cdot$$

B.8 Example 8: $\int_0^\infty G_{11}^{11}\left(t \,\middle|\, \begin{matrix}\frac{5}{3}\\0\end{matrix}\right) e^{-xt}dt$

Finally consider computing the integral

$$I(x) = \int_0^\infty G_{11}^{11}\left(t \,\middle|\, \begin{matrix}\frac{5}{3}\\0\end{matrix}\right) e^{-xt}dt.$$

Therefore we have

$$f(x) = G_{1,1}^{1,1}\left(x \,\middle|\, \begin{matrix}\frac{5}{3}\\0\end{matrix}\right) \text{ and } g(x) = e^{-x}.$$

1. (a) 0
   (b) ∞
2. (a) $\frac{4}{3}\sqrt{\pi}$
   (b) $\frac{\frac{4}{3}\sqrt{\pi}e^{(-i)t\Im(x)}\left(\frac{1}{e^t}\right)^{\Re(x)}}{\frac{1}{t}^{\frac{3}{2}}}$
3. $\Re(x) \geq 0$
4. $\Gamma\left(-\frac{3}{2} - s\right)\Gamma(s)$
5. $\frac{2\pi e^{\frac{1}{4}\pi(3i+4ix-4y)}\sqrt{\frac{1}{y}}}{y^2}$ as $y \to \infty$ and $\frac{2\pi e^{\frac{1}{4}\pi(-3i+(-4i)x+4y)}\sqrt{-\frac{1}{y}}}{y^2}$ as $y \to -\infty$
6. $-\pi < \arg(x) < \pi$
7. $\Gamma(s)$
8. $1 - i\sqrt{\pi}y^{iy}e^{\frac{1}{2}i\pi x+(-i)y-\frac{1}{2}\pi y}\left(\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y \to \infty$ and
   $1 + i(-y)^{iy}e^{(-i)y+\left(-\frac{1}{2}i\right)\pi x+\frac{1}{2}\pi y}\sqrt{\pi}\left(-\frac{1}{y}\right)^{\frac{1}{2}-x}$ as $y \to -\infty$

9. $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$

10. $\Gamma\left(-\frac{5}{2}+s\right)\Gamma\left(1-s\right)\Gamma\left(s\right)$ where $-\frac{3\pi}{2} < \arg(x) < \frac{3\pi}{2}$

11. $(2x^2+2x)y'(x)+(5+2x)y(x)=0$

12. $y'(x)+y(x)=0$

13. $(5-2s)u(s)-2su(s+1)=0$

14. $u(s+1)-su(s)=0$

15. $(-5+2s)u(s)+2u(s+1)=0$

16. $(2x-5)y(x)-2xy'(x)=\frac{8}{3}\sqrt{\pi}$

17. $2y(x)+(-7+2x)y'(x)-2xy''(x)=0$

18. 0 is a regular singularity and ∞ is an irregular singularity

19. $\frac{\frac{4}{3}\sqrt{\pi x}}{x^{\frac{3}{2}}}+\frac{2\sqrt{\pi x}}{x^{\frac{5}{2}}}-\frac{e^x\pi\operatorname{erf}(\sqrt{x})}{x^{\frac{5}{2}}}+\frac{e^x c_1}{x^{\frac{5}{2}}}$

20. $\frac{c_1}{x^{\frac{5}{2}}}+O\left(x^{-\frac{3}{2}}\right)$

21. $\frac{\pi}{x^{\frac{5}{2}}}$

22. $c_1 = Pi$

23. $\Re(x) \geq 0$

24. Omit

25. Omit

26. Omit

27. Omit

28. $I(x)=\frac{\frac{4}{3}\sqrt{\pi x}}{x^{\frac{3}{2}}}+\frac{2\sqrt{\pi x}}{x^{\frac{5}{2}}}-\frac{e^x\pi\operatorname{erf}(\sqrt{x})}{x^{\frac{5}{2}}}+\frac{e^x\pi}{x^{\frac{5}{2}}}$ for $\Re(x) \geq 0$

References

[1] Coddington E. A. and Levinson N. *Theory of Ordinary Differential Equations.* McGraw-Hill Book Company, New York, 1955.

[2] Prudnikov A.P., Brychkov Yu.A., and Marichev O.I. *Integrals and Series. Vol. 1: Elementary functions.* Gordon and Breach Sci. Publ., N. Y., London, Tokyo, 1986.

[3] Prudnikov A.P., Brychkov Yu.A., and Marichev O.I. *Integrals and Series. Vol. 2: Special functions.* Gordon and Breach Sci. Publ., N. Y., London, Tokyo, 1986.

[4] Prudnikov A.P., Brychkov Yu.A., and Marichev O.I. *Integrals and Series. Vol. 3: More special functions.* Gordon and Breach Sci. Publ., N. Y., London, Tokyo, 1989.

[5] Prudnikov A.P., Brychkov Yu.A., and Marichev O.I. *Integrals and Series. Vol. 4: Laplace transforms.* Gordon and Breach Sci. Publ., New York, 1992.

[6] N. Bleistein and N. Handelsman. *Asymptotic Expansions of Integrals.* Dover, New York, 1986.

[7] Frédéric Chyzak. Holonomic systems and automatic proofs of identities. Research Report 2371, Institut National de Recherche en Informatique et en Automatique, 1994.

[8] Frédéric Chyzak. *Fonctions holonomes en calcul formel.* Thèse universitaire, École polytechnique, 1998. INRIA, TU 0531. 227 pages.

[9] Frédéric Chyzak. Groebner bases, symbolic summation and symbolic integration. In B. Buchberger and F. Winkler, editors, *Groebner Bases and Applications (Proc. of the Conference 33 Years of Gröbner Bases)*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 32–60. Cambridge University Press, 1998.

[10] Frédéric Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Mathematics*, 217(1-3):115–134, 2000.

[11] Frédéric Chyzak and Bruno Salvy. Non-commutative elimination in Ore algebras proves multivariate holonomic identities. *Journal of Symbolic Computation*, 26(2):187–227, 1998.

[12] Keith O. Geddes, M. Lawrence Glasser, Reg A. Moore, and Tony C. Scott. Evaluation of classes of definite integrals involving elementary functions via differentiation of special functions. *Appl. Algebra Eng. Commun. Comput.*, 1:149–165, 1990.

[13] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press Inc, New York, 1980.

[14] R.B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals.* Cambridge University Press, Cambridge, 2001.

[15] Kelly Roach. Meijer G function representations. In *ISSAC '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, pages 205–211, New York, NY, USA, 1997. ACM.

[16] Bruno Salvy. A symbolic algorithm for the computation of convolution integrals, 2000.

[17] Bruno Salvy and Paul Zimmerman. GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Trans. Math. Softw.*, 20(2):163–177, 1994.

[18] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals.* Clarendon Press, Oxford, 1937.

[19] Doron Zeilberger. A fast algorithm for proving terminating hypergeometric identities. *Discrete Mathematics*, 80(2):207–211, 1990.

[20] Doron Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32(3):321–368, 1990.