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A homotopy-theoretic universal property of Leinster's operad for weak ω-categories

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Abstract

We explain how any cofibrantly generated weak factorisation system on a category may be equipped with a universally and canonically determined choice of cofibrant replacement. We then apply this to the theory of weak ω-categories, showing that the universal and canonical cofibrant replacement of the operad for strict ω-categories is precisely Leinster's operad for weak ω-categories.

1. *Introduction*

One of the most interesting aspects of modern homotopy theory is the general machinery it provides for replacing some piece of algebraic structure with a "weakened" version of that same structure. The picture is as follows: we begin with a category $\mathcal{C}$ equipped with a notion of higher-dimensionality coming from a model structure in the sense of Quillen [**14**]. We now contemplate some notion of algebraic theory on $\mathcal{C}$: monads, operads, or Lawvere theories on $\mathcal{C}$, for example. These algebraic theories themselves form a category $\mathbf{Th}(\mathcal{C})$, and by making use of various transfer techniques we obtain a model structure on $\mathbf{Th}(\mathcal{C})$ from the one on $\mathcal{C}$. Now, for a particular algebraic theory $T \in \mathbf{Th}(\mathcal{C})$, we obtain a weakened version of this theory by taking a cofibrant replacement for T in the category $\mathbf{Th}(\mathcal{C})$. A cofibrant replacement is a generalised projective resolution: and so the effect this has is to transform each algebraic law satisfied by the theory T into a piece of higher-dimensional data witnessing the weak satisfaction of that same law, with all this extra data fitting together in a coherent way.

The remarkable thing about this machinery is how little it requires to get going. All we need is a category $\mathcal{C}$, a notion of algebraic structure, and a notion of higher-dimensionality; and for this last, we do not even need a full model structure on $\mathcal{C}$. A single weak factorisation system [**4**] will do, and for a sufficiently well behaved (typically, locally presentable) $\mathcal{C}$ we may obtain this by using the small object argument of Quillen [**14**] and Bousfield [**4**]: for which it suffices to specify a set of generating higher-dimensional cells in $\mathcal{C}$, together with their boundaries and the inclusions of the latter into the former. Moreover, it is frequently the case that $\mathcal{C} = [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ for some Reedy category $\mathcal{D}$ [**9**, **16**], in which case we have canonical notions of both *cell* (the representable presheaves) and *boundary* (arising from the Reedy structure).

Yet this rather appealing construction has a problem, which arises when we ask what "the" weakened version of a particular algebraic theory T is. Because cofibrant replacements need not be unique, even up to isomorphism, we may only legitimately talk of "a" weakened version of T; and so it becomes pertinent to ask which one we choose. The usual answer given is that we don't really care, since all the choices are essentially equivalent: but since the point of being algebraic is in some sense to "pin down everything that can be pinned down", it seems perverse that we should be so hazy on this particular point.

The obvious solution is to make a definite choice of cofibrant replacements in $\mathbf{Th}(\mathcal{C})$: and if – as is almost always the case – the weak factorisation system under consideration was constructed using the small object argument, then it may be equipped with such a choice. Yet the situation is not entirely satisfactory for two reasons. Firstly, the cofibrant replacements we obtain are in no way canonical, since the induction which constructs them is governed by some (sufficiently large) regular cardinal α, with different choices of α leading to different cofibrant replacements. Secondly and more importantly, the cofibrant replacements we obtain are neither particularly natural nor computationally tractable. In principle, it would be possible to reason about them by induction; but in practice, this would require some rather strange combinatorics of a nature entirely orthogonal to that of the mathematics one was trying to do.

However, recent work of Grandis and Tholen [**8**] and the author [**7**] suggests a solution to this problem. Using the results of [**7**], we may equip any reasonable (which is to say, cofibrantly generated) weak factorisation system with a canonical and universal notion of cofibrant replacement. The canonicity says that we need specify no additional information beyond the set of generating cells and boundaries; whilst the universality tells us that the cofibrant replacements we obtain are rather natural, and in particular admit a straightforward calculus of inductive reasoning.

In this paper, we first explain the technology behind these universal cofibrant replacements, and then illustrate their utility by means of an example drawn from the study of weak ω-categories. More specifically, we consider Batanin's theory of globular operads [**1**] and, by using the machinery outlined above, obtain a canonical and universal notion of cofibrant replacement for globular operads. We then show that applying this to the globular operad for strict ω-categories yields precisely the operad singled out by Leinster [**13**] as the operad for weak ω-categories.

2. *Weak factorisation and cofibrant replacement*

2·1. *Weak factorisation systems*

A *weak factorisation system* [**4**] $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ is given by two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{C}$ which are each closed under retracts when viewed as full subcategories of the arrow category $\mathcal{C}^{\mathbf{2}}$, and which satisfy the two axioms of

(i) *factorisation*: each $f \in \mathcal{C}$ may be written as $f = pi$ where $i \in \mathcal{L}$ and $p \in \mathcal{R}$; and
(ii) *weak orthogonality*: for each $i \in \mathcal{L}$ and $p \in \mathcal{R}$, we have $i \pitchfork p$,

where to say that $i \pitchfork p$ holds is to say that for each commutative square

$$\begin{array}{ccc} U & \xrightarrow{f} & W \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ V & \xrightarrow[g]{} & X \end{array} \qquad (\star)$$

we may find a filler $j\colon V \to W$ satisfying $ji = f$ and $pj = g$. For those weak factorisation systems that we will be considering, the following terminology will be appropriate: the maps in $\mathcal{L}$ we call *cofibrations*, and the maps in $\mathcal{R}$, *acyclic fibrations*. Supposing $\mathcal{C}$ to have an initial object 0, we say that $U \in \mathcal{C}$ is *cofibrant* just when the unique map $0 \to U$ is a cofibration; and define a *cofibrant replacement* for $X \in \mathcal{C}$ to be a factorisation of the unique map $0 \to X$ as a cofibration followed by an acyclic fibration:

$$0 \longrightarrow X' \xrightarrow{p} X.$$

The principal tool we use for the construction of weak factorisation systems is the following result, first proved by Quillen in the finitary case [**14**, chapter II, Section 3] and in the following transfinite form by Bousfield [**4**]. For a modern account, see [**10**], for example.

PROPOSITION 1 (The small object argument). *Let $\mathcal{C}$ be a locally presentable category, and let I be a set of maps in $\mathcal{C}$. Define classes of maps $I^{\downarrow}$ and $I^{\downarrow\uparrow}$ by*

$$I^{\downarrow} := \left\{ p \in \mathcal{C}^{\mathbf{2}} \,\middle|\, j \pitchfork p \text{ for all } j \in I \right\}$$

and

$$I^{\downarrow\uparrow} := \left\{ i \in \mathcal{C}^{\mathbf{2}} \,\middle|\, i \pitchfork p \text{ for all } p \in I^{\downarrow} \right\}.$$

Then the pair $(I^{\downarrow\uparrow}, I^{\downarrow})$ is a weak factorisation system on $\mathcal{C}$.

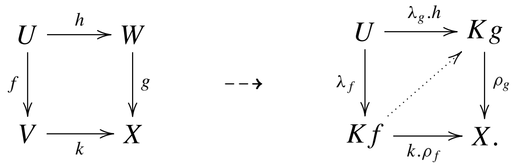
We call I the set of *generating cofibrations* for $(I^{\downarrow\uparrow}, I^{\downarrow})$, and given a map $i\colon U \to V$ in I, we call V a *generating cell* and U its *boundary*. To say that a weak factorisation system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* is to say that there is some set I for which $(\mathcal{L}, \mathcal{R}) = (I^{\downarrow\uparrow}, I^{\downarrow})$. Observe that this I will usually not be unique; however, this is not a problem since we typically begin with the set I and generate the weak factorisation system from it, rather than vice versa.

2·2. *Functorial w.f.s.'s*

Given a w.f.s. $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$, it may be the case that for each morphism $f\colon X \to Y$ of $\mathcal{C}$, we can provide a choice

$$X \xrightarrow{\lambda_f} Kf \xrightarrow{\rho_f} Y$$

of $(\mathcal{L}, \mathcal{R})$ factorisation for f. Suppose this is so; then by weak orthogonality, we know that for each square as on the left of the following diagram, there exists a filler for the corresponding square on the right:



It may now be that we can choose a diagonal filler $K(h,k)\colon Kf \to Kg$ for each such square: and that we can do so in such a way that the assignations $f \mapsto Kf$ and $(h,k) \mapsto K(h,k)$ underlie a functor $K\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. If this is so, then the maps λ_f and ρ_f necessarily provide the components of natural transformations $\lambda\colon \mathrm{cod} \Rightarrow K$ and $\rho\colon K \Rightarrow \mathrm{dom}$; and we call the triple (K, λ, ρ) so obtained a *functorial realisation* of $(\mathcal{L}, \mathcal{R})$.

PROPOSITION 2. *Let $\mathcal{C}$ be a locally presentable category, and let I be a set of maps in $\mathcal{C}$. Then for each choice of a sufficiently large regular cardinal α, the small object argument provides us with a functorial realisation $(K^{(\alpha)}, \lambda^{(\alpha)}, \rho^{(\alpha)})$ of the weak factorisation system $(I^{\downarrow\uparrow}, I^{\downarrow})$.*

The proof falls out of the construction used in the small object argument. The regular cardinal α that we provide serves to fix the length of the transfinite induction by which factorisations are constructed. Note that the functorial realisation we obtain depends not only upon α but also upon the particular set I of generating cofibrations that we choose.

Remark 3. It was shown in [**17**, section 2·4] that the data (K, λ, ρ) for a functorial realisation completely determines the underlying w.f.s. $(\mathcal{L}, \mathcal{R})$. To see this, we define two auxiliary functors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ whose action on objects is given by

$$L\left(\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}\right) = \begin{array}{c} X \\ \downarrow{\scriptstyle \lambda_f} \\ Kf \end{array} \qquad \text{and} \qquad R\left(\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}\right) = \begin{array}{c} Kf \\ \downarrow{\scriptstyle \rho_f} \\ Y; \end{array}$$

and two auxiliary natural transformations $\Lambda\colon \mathrm{id}_{\mathcal{C}^2} \Rightarrow R$ and $\Phi\colon L \Rightarrow \mathrm{id}_{\mathcal{C}^2}$ whose respective components at $f\colon X \to Y$ are:

$$\Lambda_f = \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \rho_f} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array} \qquad \text{and} \qquad \Phi_f = \begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle f} \\ Kf & \xrightarrow[\rho_f]{} & Y. \end{array}$$

We may now show that a morphism $f\colon X \to Y$ of $\mathcal{C}$ lies in $\mathcal{R}$ just when the map $\Lambda_f\colon f \to Rf$ admits a retraction in $\mathcal{C}^{\mathbf{2}}$: which is to say that f is an algebra for the pointed endofunctor (R, Λ). Dually, we may show that f lies in $\mathcal{L}$ just when the map $\Phi_f\colon Lf \to f$ admits a section: which is to say that f is a coalgebra for the copointed endofunctor (L, Φ). As a particular case of this last fact, if $\mathcal{C}$ has an initial object, then $L\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ restricts and corestricts to the coslice $0/\mathcal{C} \cong \mathcal{C}$ to yield a *cofibrant replacement functor* $Q\colon \mathcal{C} \to \mathcal{C}$ together with a copointing $\epsilon\colon Q \Rightarrow \mathrm{id}_{\mathcal{C}}$; and now an an object $X \in \mathcal{C}$ is cofibrant just when it may be made into a coalgebra for (Q, ϵ); which is to say, just when $\epsilon_X\colon QX \to X$ admits a section in $\mathcal{C}$.

2·3. *Natural w.f.s.'s*

As we mentioned in the Introduction, the functorial realisations $(K^{(\alpha)}, \lambda^{(\alpha)}, \rho^{(\alpha)})$ that we obtain from the small object argument are not very intuitive. One way of rectifying this is through a further strengthening of the notion of weak factorisation system. We begin from a w.f.s. $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ together with a functorial realisation (K, λ, ρ) thereof. Now, since in any w.f.s. the classes of $\mathcal{L}$-maps and $\mathcal{R}$-maps are closed under composition, we have fillers for squares of the following form:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{\lambda_f}} & K\lambda_f \\ {\scriptstyle \lambda_f}\downarrow & \nearrow & \downarrow{\scriptstyle \rho_f \cdot \rho_{\lambda_f}} \\ Kf & \xrightarrow[\rho_f]{} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ {\scriptstyle \lambda_{\rho_f} \cdot \lambda_f}\downarrow & \nearrow & \downarrow{\scriptstyle \rho_f} \\ K\rho_f & \xrightarrow[\rho_{\rho_f}]{} & Y, \end{array}$$

and it may be the case that we can provide a choice of fillers $\sigma_f\colon Kf \to K\lambda_f$ and $\pi_f\colon K\rho_f \to Kf$ for each such square; and that we can do so in such a way that the morphisms $\Sigma_f\colon Lf \to LLf$ and $\Pi_f\colon RRf \to Rf$ of $\mathcal{C}^{\mathbf{2}}$ given by

$$\Sigma_f = \begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle \lambda_{\lambda_f}} \\ Kf & \xrightarrow[\sigma_f]{} & K\lambda_f \end{array} \quad \text{and} \quad \Pi_f = \begin{array}{ccc} K\rho_f & \xrightarrow{\pi_f} & Kf \\ {\scriptstyle \rho_{\rho_f}}\downarrow & & \downarrow{\scriptstyle \rho_f} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

provide the components at f of natural transformations $\Sigma\colon L \Rightarrow LL$ and $\Pi\colon RR \Rightarrow R$. Under these circumstances, it may be that $\mathsf{R} = (R, \Lambda, \Pi)$ describes a monad on $\mathcal{C}^{\mathbf{2}}$, and $\mathsf{L} = (L, \Phi, \Sigma)$ a comonad; and that the natural transformation $\Delta\colon LR \Rightarrow RL\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ with components

$$\Delta_f = \begin{array}{ccc} Kf & \xrightarrow{\sigma_f} & X \\ {\scriptstyle \lambda_{\rho_f}}\downarrow & & \downarrow{\scriptstyle \rho_{\lambda_f}} \\ K\rho_f & \xrightarrow[\pi_f]{} & Kf \end{array}$$

describes a distributive law [**3**] between L and R. Under these circumstances, we will say that (L, R) is an *algebraic realisation* of $(\mathcal{L}, \mathcal{R})$. The pairs (L, R) arising in this way are the *natural weak factorisation systems* of [**8**]. Though the requirements for an algebraic realisation may appear strong, they are in fact rather easily satisfied:

PROPOSITION 4 (The refined small object argument). *Let $\mathcal{C}$ be a locally presentable category, and let I be a set of maps in $\mathcal{C}$. Then the weak factorisation system $(I^{\downarrow\uparrow}, I^{\downarrow})$ has an algebraic realisation* (L, R).

Proof. For a full proof see [**7**, theorem 4·4]: we recall only the salient details here. We begin exactly as in the small object argument. Given a map $f\colon X \to Y$ of $\mathcal{C}$ we consider the set

$$S := \left\{ (j, h, k) \,\middle|\, j : A \to B \in I, (h, k)\colon j \to f \in \mathcal{C}^{\mathbf{2}} \right\}.$$

We have a commutative diagram

$$\begin{array}{ccc} \sum_{x\in S} A_x & \xrightarrow{\langle h_x\rangle_{x\in S}} & X \\ {\scriptstyle \sum_{x\in S} j_x}\downarrow & & \downarrow{\scriptstyle f} \\ \sum_{x\in S} B_x & \xrightarrow[\langle h_x\rangle_{x\in S}]{} & Y \end{array}$$

in $\mathcal{C}$; and may factorise this as

$$\begin{array}{ccccc} \sum_{x\in S} A_x & \xrightarrow{\langle h_x\rangle_{x\in S}} & X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \sum_{x\in S} j_x}\downarrow & & \downarrow{\scriptstyle \lambda'_f} & & \downarrow{\scriptstyle f} \\ \sum_{x\in S} B_x & \longrightarrow & K'f & \xrightarrow[\rho'_f]{} & Y \end{array}$$

where the left-hand square is a pushout. The assignation $f \mapsto \rho'_f$ may now be extended to a functor $R'\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$; whereupon the map $(\lambda'_f, \mathrm{id}_Y)\colon f \to R'f$ provides the component at f of a natural transformation $\Lambda' \colon \mathrm{id}_{\mathcal{C}^2} \Rightarrow R'$. We now obtain the monad part R of the desired algebraic realisation as the free monad on the pointed endofunctor (R', Λ'). We may construct this using the techniques of [**12**]. To obtain the comonad part L we proceed as follows. The assignation $f \mapsto \lambda'_f$ underlies a functor $L' \colon \mathcal{C}^2 \to \mathcal{C}^2$; and a little manipulation shows that this functor in turn underlies a comonad L' on $\mathcal{C}^2$. We may now adapt the free monad construction so that at the same time as it produces R from (R', Λ'), it also produces L from L'.

The algebraic realisation of $(I^{\downarrow\uparrow}, I^{\downarrow})$ given in Proposition 4 satisfies a universal property with respect to I which determines it up to unique isomorphism. Thus we refer to (L, R) as the *universal algebraic realisation of* I. To give its universal property, we consider algebras for the monad R. From Remark 3, we know that acyclic fibrations coincide with algebras for the pointed endofunctor (R, Λ): and so algebras for the monad R must be acyclic fibrations equipped with certain extra data. The following Proposition makes this precise.

PROPOSITION 5. *Let* (L, R) *be the universal algebraic realisation of a set of maps* I *as given in Proposition* 4. *To give an algebra for the monad* R *on* $\mathcal{C}^2$ *is to give a map* $p \colon W \to X$ *of* $\mathcal{C}$ *together with, for each* $i \colon U \to V$ *in* I *and commutative square*

$$\begin{array}{ccc} U & \xrightarrow{f} & W \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ V & \xrightarrow[g]{} & X \end{array}$$

a choice of diagonal filler $j \colon V \to W$, *subject to no further conditions, whilst to give a morphism of* R*-algebras is to give a map of* $\mathcal{C}^2$ *which strictly commutes with the chosen liftings. Moreover, this characterisation of the category of* R*-algebras determines the pair* (L, R) *up to unique isomorphism.*

Proof. See [**7**, proposition 5·4].

Dually, we may think of coalgebras for the comonad L as cofibrations equipped with extra data. There is much less that can be said about these at a general level: however, a good intuition is that if the cofibrations are retracts of relatively free things then the L-coalgebras are the relatively free things of which they are retracts.[1]

Remark 6. Observe that for any algebraically realised w.f.s. (L, R) on a category with initial object, the chosen cofibrant replacements underlie a *cofibrant replacement comonad* $\mathsf{Q} = (Q, \epsilon, \delta)$ which is the restriction and corestriction of L to the coslice under 0. The concept of a cofibrant replacement comonad was first considered by [**15**], though it should be noted that the comonads constructed there do not coincide with the ones obtained from Proposition 4. Indeed, they are built using the small object argument, and so suffer from the same dependence upon a regular cardinal α that we noted in Proposition 2.

Example 7. Consider the category $\mathbf{Ch}(R)$ of positively graded chain complexes of R-modules, equipped with the set of generating cofibrations $I := \{\, \partial y(i) \hookrightarrow y(i) \mid i \in \mathbb{N} \,\}$.

[1] The problem of ascertaining circumstances under which this intuition is valid is closely related to the following old problem: given an adjunction which is known to be monadic, under which circumstances is it also comonadic?

Here $y(i)$ is the representable chain complex at i, with components given by

$$y(i)_n = \begin{cases} R & \text{if } n = i \text{ or } n = i - 1; \\ 0 & \text{otherwise,} \end{cases}$$

and as differential, the identity map $R \to R$ at stage i and the zero map elsewhere. The chain complex $\partial y(i)$ is its boundary, whose components are

$$\partial y(i)_n = \begin{cases} R & \text{if } n = i - 1; \\ 0 & \text{otherwise,} \end{cases}$$

and whose differential is everywhere zero. Since $\mathbf{Ch}(R)$ is locally finitely presentable, we may apply Proposition 4 to obtain an algebraically realised w.f.s. (L, R). We describe the cofibrant replacement $\epsilon_X \colon QX \to X$ that this provides for $X \in \mathbf{Ch}(R)$. The chain complex QX will be free in every dimension; and so it suffices to specify a set of free generators for each $(QX)_i$ and to specify where each generator should be sent by the differential $d'_i \colon (QX)_i \to (QX)_{i-1}$ and the counit $\epsilon_i \colon (QX)_i \to X_i$. We do this by induction over i:

(i) for the base step, $(QX)_0$ is generated by the set $\{\, x \mid x \in X_0 \,\}$, and $\epsilon_0 \colon (QX)_0 \to X_0$ is specified by $\epsilon_0(x) = x$ and $d'_0 \colon (QX)_0 \to 0$ is the zero map;

(ii) for the inductive step, $(QX)_{i+1}$ (for $i \geqslant 0$) is generated by the set

$$\left\{\, (x, z) \,\middle|\, x \in X_{i+1},\ z \in \ker d'_i,\ \epsilon_i(z) = d_{i+1}(x) \,\right\},$$

whilst $\epsilon_{i+1} \colon (QX)_{i+1} \to X_{i+1}$ and $d'_{i+1} \colon (QX)_{i+1} \to (QX)_i$ are specified by

$$\epsilon_{i+1}(x, z) = x \qquad \text{and} \qquad d'_{i+1}(x, z) = z.$$

Note that, in particular, we may view any R-module M as a chain complex concentrated in degree 0; whereupon the above construction reduces to the usual bar resolution of M. We can characterise a typical Q-coalgebra as being given by a chain complex X equipped with, for each $i \in \mathbb{N}$, a subset $G_i \subset X_i$ for which the inclusion map $G_i \hookrightarrow X_i$ exhibits X_i as the free R-module on G_i.

2·4. *Constructions on w.f.s.'s*

We end this section by reviewing two standard techniques for transferring w.f.s.'s between categories that we shall need in the sequel. In both cases, we assume the category $\mathcal{C}$ is locally presentable, so that we may freely apply Proposition 4. For the first transfer technique, we consider passage to the slice.

PROPOSITION 8. *If $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system on $\mathcal{C}$, and $X \in \mathcal{C}$, then there is an induced weak factorisation system $(\mathcal{L}', \mathcal{R}')$ on $\mathcal{C}/X$ for which $\mathcal{L}'$ and $\mathcal{R}'$ are the preimages of $\mathcal{L}$ and $\mathcal{R}$ under the forgetful functor $U \colon \mathcal{C}/X \to \mathcal{C}$. If I is a set which cofibrantly generates $(\mathcal{L}, \mathcal{R})$, then the set I' of preimages of I under U generates $(\mathcal{L}', \mathcal{R}')$; and if we let* (L, R) *and* $(\mathsf{L}', \mathsf{R}')$ *denote the universal algebraic realisations of I and I', then there is a functor $\tilde{U} \colon$* R'-$\mathbf{Alg} \to \mathsf{R}$-$\mathbf{Alg}$ *making the following diagram a pullback*:

$$\begin{array}{ccc} \mathsf{R}'\text{-}\mathbf{Alg} & \xrightarrow{\tilde{U}} & \mathsf{R}\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ (\mathcal{C}/X)^2 & \xrightarrow[U^2]{} & \mathcal{C}^2. \end{array}$$

Proof. Mostly trivial; the final part follows from the characterisation of R-**Alg** given in Proposition 5.

Our second transfer technique allows us to lift a cofibrantly generated weak factorisation system along a right adjoint functor. This process was first described in the general context of model categories by Sjoerd Crans [**6**].

PROPOSITION 9. *Let* $(\mathcal{L}, \mathcal{R})$ *be a cofibrantly generated w.f.s. on* $\mathcal{C}$*, and suppose that* $F \dashv G \colon \mathcal{D} \to \mathcal{C}$ *with* $\mathcal{D}$ *locally presentable. Then there is a w.f.s.* $(\mathcal{L}', \mathcal{R}')$ *on* $\mathcal{D}$ *for which* $\mathcal{R}'$ *is the preimage of* $\mathcal{R}$ *under* G*. Moreover, if* I *is a generating set for* $(\mathcal{L}, \mathcal{R})$*, then* $I' = \{ Fi \mid i \in I \}$ *is a generating set for* $(\mathcal{L}', R')$*; and if we let* (L, R) *and* $(\mathsf{L}', \mathsf{R}')$ *denote the universal algebraic realisations of* I *and* I'*, then there is a functor* $\tilde{G} \colon \mathsf{R}'\text{-}\mathbf{Alg} \to \mathsf{R}\text{-}\mathbf{Alg}$ *making the following diagram a pullback:*

$$\begin{array}{ccc} \mathsf{R}'\text{-}\mathbf{Alg} & \xrightarrow{\tilde{G}} & \mathsf{R}\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathbf{2}} & \xrightarrow[G^{2}]{} & \mathcal{C}^{\mathbf{2}}. \end{array}$$

Proof. The key observation is that there is a bijection between fillers for diagrams of the forms

$$\begin{array}{ccc} U & \xrightarrow{f} & GW \\ {\scriptstyle i}\downarrow & \overset{j}{\nearrow} & \downarrow{\scriptstyle Gp} \\ V & \xrightarrow[g]{} & GX \end{array} \qquad \text{and} \qquad \begin{array}{ccc} FU & \xrightarrow{\bar{f}} & W \\ {\scriptstyle Fi}\downarrow & \overset{\bar{j}}{\nearrow} & \downarrow{\scriptstyle p} \\ FV & \xrightarrow[\bar{g}]{} & X, \end{array}$$

where $\bar{f}$, $\bar{g}$ and $\bar{j}$ denote the transposes of f, g and j under adjunction. The remaining details are straightforward.

3. *Application to the theory of weak ω-categories*

3·1. *The goal*

By combining the material of the previous section with the techniques outlined in the Introduction, we obtain a machinery that can weaken algebraic structures in a canonical way. In this section, we will use this in the context of Batanin's theory of weak ω-categories [**1**] to show that the canonical weakening of the theory of strict ω-categories is precisely the theory of weak ω-categories singled out by Leinster in [**13**]. Such a result is strongly hinted at in the work of Batanin and Leinster (see in particular the remarks following [**1**, lemma 8·1]), but is never spelt out in detail; and so our result serves as a clarification of the relationship between weak ω-categories and other kinds of weak algebraic structure.

3·2. *The ingredients*

Recall that the key ingredients required for the machinery of the Introduction are a base category $\mathcal{C}$; a notion of higher-dimensionality on $\mathcal{C}$ arising from a weak factorisation system; a category $\mathbf{Th}(\mathcal{C})$ of theories on $\mathcal{C}$; and a particular theory $T \in \mathbf{Th}(\mathcal{C})$ we wish to weaken. We now describe each of these four ingredients for our example.

3·2·1. *The base category*

Our base category will be the category **GSet** of *globular sets*. This is the category $[\mathbb{G}^{op}, \mathbf{Set}]$ of presheaves over the *globe category* $\mathbb{G}$, which in turn may be presented as the free category on the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots$$

subject to the *coglobularity equations* $\sigma_{n+1}\sigma_n = \tau_{n+1}\sigma_n$ and $\sigma_{n+1}\tau_n = \tau_{n+1}\tau_n$ for all n. Thus a globular set $X \in \mathbf{GSet}$ is given by sets X_n of *n-cells* together with source and target functions $s_n, t_n \colon X_{n+1} \to X_n$ subject to the *globularity equations*, which assert that the source and target n-cells of any $(n+1)$-cell are parallel.

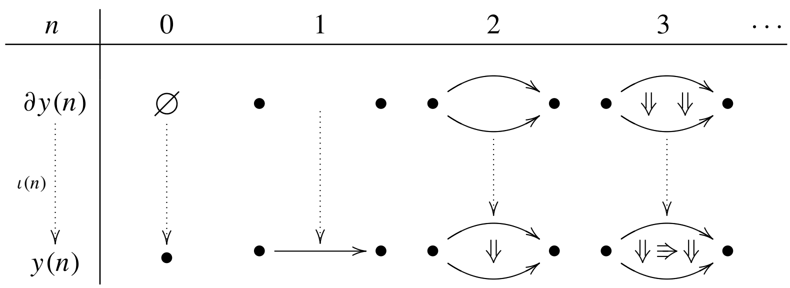
3·2·2. *The weak factorisation system*

Our notion of higher-dimensionality on **GSet** will be obtained by a *Reedy category* technique [**16**]. The definition of a Reedy category is quite subtle—see [**9**, chapter 15] for instance—but we will not need the full generality here. Rather, we consider the simpler notion of a *direct category*; this being a small category $\mathcal{A}$ which admits an identity-reflecting functor $\dim \colon \mathcal{A} \to \gamma$ for some ordinal γ. For such a category $\mathcal{A}$, the presheaf category $\hat{\mathcal{A}} := [\mathcal{A}^{op}, \mathbf{Set}]$ comes equipped with canonical notions of *generating cell* and *boundary*. The generating cells are the representable presheaves $y(a) := \mathcal{A}(-, a)$; whilst their boundaries $\partial y(a)$ are given by the coend

$$\partial y(a) := \int^{\substack{b \in \mathcal{A} \\ \dim(b) < \dim(a)}} \mathcal{A}(b, a) \cdot y(b)$$

in $\hat{\mathcal{A}}$. The universal property of the displayed coend together with the Yoneda lemma induces a canonical map of presheaves $\iota(a) \colon \partial y(a) \to y(a)$; and so we obtain a set of generating cofibrations $I := \{\, \iota(a) \mid a \in \mathcal{A}\,\}$. Since any presheaf category is locally finitely presentable, we may apply Proposition 4 to obtain an algebraically realised w.f.s. on $\hat{\mathcal{A}}$ generated by the set I.

The category $\mathbb{G}$ is a direct category, with $\gamma = \omega$ and dim the unique identity-on-objects functor $\mathbb{G} \to \omega$; and so applying the theory of the previous paragraph yields the following set of generating cofibrations in **GSet**:



We may describe the presheaves $\partial y(n)$ explicitly as follows. We have that $\partial y(0) = 0$ and $\partial y(1) = y(0) + y(0)$; whilst each subsequent $\partial y(n+2)$ is obtained as the pushout

$$\begin{array}{ccc} y(n) + y(n) & \xrightarrow{[y(\sigma_n),\ y(\tau_n)]} & y(n+1) \\ {\scriptstyle [y(\sigma_n),\ y(\tau_n)]}\downarrow & & \downarrow \\ y(n+1) & \longrightarrow & \partial y(n+2). \end{array}$$

The inclusion maps $\iota(n)\colon \partial y(n) \to y(n)$ are given by taking $\iota(0)\colon 0 \to y(0)$ to be the unique map; taking $\iota(1)\colon y(0) + y(0) \to y(1)$ to be $[y(\sigma_0),\ y(\tau_0)]$; and taking each subsequent $\iota(n+2)\colon \partial y(n+2) \to y(n+2)$ to be the map induced using the universal property of pushout with respect to the commutative square

$$\begin{array}{ccc} y(n) + y(n) & \xrightarrow{[y(\sigma_n),\ y(\tau_n)]} & y(n+1) \\ {\scriptstyle [y(\sigma_n),\ y(\tau_n)]}\downarrow & & \downarrow{\scriptstyle y(\sigma_{n+1})} \\ y(n+1) & \xrightarrow[y(\tau_{n+1})]{} & y(n+2). \end{array}$$

3·2·3. *The category of theories*

We now give our notion of theory on **GSet**. These will be Batanin's *globular operads*, which were introduced in [**1**]; though our presentation of them will follow that given in [**13**]. We may see globular operads as a generalisation of **Set**-based operads. Recall that we specify a **Set**-based operad by giving a collection $\{\, \mathcal{O}(n) \mid n \in \mathbb{N} \,\}$ of basic n-ary operations, together with data expressing how these operations compose together. The collection of basic n-ary operations amounts to an object $\mathcal{O}$ of the category $\mathbf{Coll} = [\mathbb{N}, \mathbf{Set}]$: and any such object induces a functor $\mathcal{O} \otimes (\text{–})\colon \mathbf{Set} \to \mathbf{Set}$ given by

$$\mathcal{O} \otimes X = \sum_n \mathcal{O}(n) \times X^n.$$

In order for this functor to underlie a monad on **Set**, we require that $\mathcal{O}$ should be a monoid with respect to the "substitution" tensor product on **Coll**, whose unit is $(0, 1, 0, 0, \dots)$, and whose binary tensor is

$$(A \otimes B)(n) = \sum_{\substack{k, n_1, \dots, n_k \\ n_1 + \cdots + n_k = n}} A(n) \times B(n_k) \times \cdots \times B(n_k).$$

We call such a monoid an *operad*; and define an algebra for an operad $\mathcal{O}$ to be an algebra for the induced monad $\mathcal{O} \otimes (\text{–})$ on **Set**. We may specify globular operads and their algebras in a similar manner. First we define the category **GColl** of collections of basic globular operations. We can present this as the slice category $\mathbf{GSet}/\mathrm{pd}$, where pd is the globular set of *pasting diagrams*. If we write $(\text{–})^*$ for the free monoid monad on **Set**, then pd may be defined inductively by

$$\mathrm{pd}_0 = \{\star\} \quad \text{and} \quad \mathrm{pd}_{n+1} = (\mathrm{pd}_n)^*,$$

with source and target maps given by $s_0 = t_0 = !\colon \mathrm{pd}_1 \to \mathrm{pd}_0$, $s_{n+1} = (s_n)^*$ and $t_{n+1} = (t_n)^*$. However, we will prefer to view **GColl** as $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$, where $\mathbb{A}$ is the category of elements of pd. Any object $\mathcal{O} \in [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ induces a functor $\mathcal{O} \otimes (\text{–})\colon \mathbf{GSet} \to \mathbf{GSet}$ given by

$$(\mathcal{O} \otimes X)_i = \sum_{\pi \in \mathrm{pd}_i} \mathcal{O}(\pi) \times \mathbf{GSet}(\hat{\pi}, X),$$

where $\hat{\pi}$ is the realisation of π as a globular set: see [**13**, section 4·2] for more details. In order for this functor to underlie a monad on **GSet**, we require that $\mathcal{O}$ should be a monoid with respect to the "substitution" tensor product on **GColl**. A description of this tensor product may be found in [**13**, section 4·3], which we do not repeat since we do not need the details. We call a monoid with respect to this tensor product a *globular operad*; and define an algebra for a globular operad $\mathcal{O}$ to be an algebra for the induced monad $\mathcal{O} \otimes (\text{–})$ on **GSet**. A globular operad morphism is just a monoid morphism in **GColl**; and so we obtain a category **GOpd** of globular operads.

However, there is a small subtlety we must deal with. Part of the data for a globular operad $\mathcal{O}$ is a set of 0-dimensional operations $\mathcal{O}(\star)$, where $\star$ is the unique element of pd_0. The operad structure of $\mathcal{O}$ descends to a monoid structure on the set $\mathcal{O}(\star)$; and an $\mathcal{O}$-algebra structure on a globular set X descends to a left action of $\mathcal{O}(\star)$ on the set of 0-cells X_0. But if a globular operad $\mathcal{O}$ is to represent a theory of weak ω-categories, then its monoid of 0-dimensional operations should be trivial, since we want the "free weak ω-category" functor to be bijective on 0-cells. In order for the general machinery to take account of this fact, we take our category of theories to be the category **NGOpd** of *normalised globular operads*; this is the full subcategory of **GOpd** whose objects are those globular operads with $\mathcal{O}(\star)$ a singleton. The restriction to normalised globular operads also plays a central role in [**2**].

3·2·4. *The candidate theory*

The fourth and final ingredient we require for our machinery is a theory $\mathcal{T} \in$ **NGOpd** which we wish to weaken. We take this to be the *terminal globular operad* $\mathcal{T}$ given by $\mathcal{T}(\pi) = 1$ for all $\pi \in \mathbb{A}$. This embodies the theory of strict ω-categories, in the sense that the corresponding monad $\mathcal{T} \otimes (\text{–})$ on **GSet** is the free strict ω-category monad.

3·3. *The transfer*

We now have all the ingredients needed for our machinery. The first stage in applying it is to transfer the notion of higher-dimensionality from **GSet** to **NGOpd**. First we transfer from **GSet** to **GColl** $\cong$ **GSet**/pd using Proposition 8. This yields a cofibrantly generated w.f.s. on **GSet**/pd, with set of generating cofibrations

$$I' := \left\{ \begin{array}{c} \partial y(n) \xrightarrow{\iota(n)} y(n) \\ {\scriptstyle \pi.\iota(n)} \searrow \quad \swarrow {\scriptstyle \pi} \\ \mathrm{pd} \end{array} \middle| \; n \in \mathbb{N},\ \pi \in \mathrm{pd}_n \right\}.$$

If we view **GColl** instead as $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$, then this set I' has an alternative description. Indeed, $\mathbb{A} = \mathrm{el}(\mathrm{pd})$ is another example of a direct category so that the technique described in Section 3·2·2 may be applied; and it is easy to check that the set $\{\, \iota(\pi) \colon \partial y(\pi) \to y(\pi) \mid \pi \in \mathbb{A} \,\}$ so obtained coincides with I'. The next step is to transfer this weak factorisation system from **GColl** to **NGOpd**. We have adjunctions

$$\mathbf{NGOpd} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{GOpd} \underset{H}{\overset{V}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{GColl}: \tag{3·1}$$

indeed, **NGOpd**, **GOpd** and **GColl** are categories of models for essentially-algebraic theories in the sense of Freyd; and both U and V are induced by forgetting essentially-algebraic structure, and so have left adjoints. Essential algebraicity also implies that **NGOpd** is locally finitely presentable, so that we may lift along VU using Proposition 9 to obtain an algebraically realised w.f.s. on **NGOpd**, with set of generating cofibrations $I'' := \{ HF(\iota(\pi)) \mid \pi \in \mathbb{A} \}$.

3·4. *The result*

We are now ready to give our main result. We write (L, R) for the universal algebraic realisation of the set of generating cofibrations I'' in **NGOpd**; and as in Remark 6, we write Q for the cofibrant replacement comonad associated with (L, R).

THEOREM 10. *Applying the cofibrant replacement comonad* Q *of* **NGOpd** *to the strict* ω*-category operad* T *yields the weak* ω*-category operad* L *defined by Leinster in* [**13**, section 4].

In order to prove this, we must first recall what Leinster's operad L is. The central notion (cf. [**13**, p. 139]) is that of a *contraction* on an object of $C \in$ **GColl**. To give this we view C as a functor $\mathbb{A}^{\mathrm{op}} \to$ **Set**; now for each $\pi \in \mathrm{pd}_1$, we define $P_\pi(C)$ to be given by the set $C(s_0(\pi)) \times C(t_0(\pi))$, whilst for each $n \geqslant 2$ and $\pi \in \mathrm{pd}_n$, we define $P_\pi(C)$ to be the pullback

$$\begin{array}{ccc} P_\pi(C) & \longrightarrow & C(s_{n-1}(\pi)) \\ \downarrow & & \downarrow{\scriptstyle (s_{n-2},t_{n-2})} \\ C(t_{n-1}(\pi)) & \xrightarrow[(s_{n-2},t_{n-2})]{} & C(s_{n-2}s_{n-1}(\pi)) \times C(t_{n-2}s_{n-1}(\pi)). \end{array}$$

A *contraction* κ on C is now given by functions $\kappa_\pi \colon P_\pi(C) \to C(\pi)$ for each $n \geqslant 1$ and $\pi \in \mathrm{pd}_n$ which render commutative the evident triangles

$$\begin{array}{ccc} P_\pi(C) & \xrightarrow{\kappa_\pi} & C(\pi) \\ & \searrow \quad \swarrow & \\ & C(t_{n-1}(\pi)) \times C(s_{n-1}(\pi)). & \end{array}$$

Any morphism $f \colon C \to D$ in **GColl** induces morphisms $P_\pi(f) \colon P_\pi(C) \to P_\pi(D)$ for each $n \geqslant 1$ and $\pi \in \mathrm{pd}_n$, so that if κ and λ are contractions on C and D respectively, we may say that f *preserves the contraction* just when $f(\pi).\kappa_\pi = \lambda_\pi . P_\pi(f)$. We now define the category **OWC** of *operads with contraction* to have:

(i) **objects** being pairs $(\mathcal{K}, \kappa)$, where $\mathcal{K} \in$ **GOpd** and κ is a contraction on $U(K)$;

(ii) **morphisms** $f \colon (\mathcal{K}, \kappa) \to (\mathcal{K}', \kappa')$ being maps $f \colon \mathcal{K} \to \mathcal{K}'$ of globular operads for which $U(f)$ preserves the contraction.

The operad L of Theorem 10 is now defined to be the underlying operad L of the initial object (L, λ) of **OWC**.

Proof of Theorem 10. First note that as well as a cofibrant replacement comonad Q, we also have an "acyclically fibrant replacement monad" P on **NGOpd**, obtained by restricting and corestricting R to the slice over $\mathcal{T}$. The object $\mathsf{Q}(\mathcal{T})$ that we are interested in is given by the universally determined factorisation of the unique map $\mathcal{I} \to \mathcal{T}$, where $\mathcal{I}$ is the initial object of **NGOpd**. But this is equally well a description of $\mathsf{P}(\mathcal{I})$. It follows that we may characterise $\mathsf{Q}(\mathcal{T})$ as the underlying normalised operad of the initial P-algebra.

Let us now use Propositions 5, 8 and 9 to give an explicit description of the category of P-algebras. Recall from Section 3·2·2 the set of maps $I = \{ \partial y(n) \to y(n) \mid n \in \mathcal{N} \}$ in **GSet**.

Let us write $(\mathsf{L}_{\mathbf{GSet}}, \mathsf{R}_{\mathbf{GSet}})$ for the corresponding universally determined algebraic realisation, and $\mathsf{P}_{\mathbf{GSet}/\mathrm{pd}}$ for the restriction and corestriction of $\mathsf{R}_{\mathbf{GSet}}$ to the slice over $\mathrm{pd} \in \mathbf{GSet}$. Propositions 8 and 9 now tell us that we have a pullback diagram

$$\begin{array}{ccc} \mathsf{P}\text{-}\mathbf{Alg} & \longrightarrow & \mathsf{P}_{\mathbf{GSet}/\mathrm{pd}}\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathbf{NGOpd} & \xrightarrow[VU]{} & \mathbf{GSet}/\mathrm{pd}; \end{array}$$

whilst Proposition 5 provides us with an explicit description of the category $\mathsf{P}_{\mathbf{GSet}/\mathrm{pd}}\text{-}\mathbf{Alg}$. Putting these facts together, we find that to give an object of $\mathsf{P}\text{-}\mathbf{Alg}$ is to give a normalised globular operad $\mathcal{C}$, together with chosen fillers for every diagram of the form

$$\begin{array}{ccc} \partial y(n) & \longrightarrow & C \\ {\scriptstyle \iota(n)}\downarrow & \nearrow & \downarrow{\scriptstyle c} \\ y(n) & \longrightarrow & \mathrm{pd}, \end{array} \tag{3·2}$$

where the arrow down the right is the underlying globular collection of $\mathcal{C}$. Using the explicit construction of the maps $\iota(n)$ given in Section 3·2·2, we see that to give chosen fillers in (3·2) is trivial when $n = 0$ (by normalisation of $\mathcal{C}$); and that for $n \geqslant 1$, it is precisely to give the functions $\kappa_\pi : P_\pi(C) \to C(\pi)$ described following the statement of Theorem 10, and so amounts to giving a contraction on C. A similar argument shows that to give a morphism of $\mathsf{P}\text{-}\mathbf{Alg}$ is to give a morphism of underlying normalised globular operads $\mathcal{C} \to \mathcal{D}$ which preserves the contraction. Thus we obtain a pullback diagram

$$\begin{array}{ccc} \mathsf{P}\text{-}\mathbf{Alg} & \xrightarrow{\tilde{U}} & \mathbf{OWC} \\ \downarrow & & \downarrow \\ \mathbf{NGOpd} & \xrightarrow[U]{} & \mathbf{GOpd}. \end{array} \tag{3·3}$$

By the remarks at the start of the proof, we will be done if we can show that $\tilde{U}$ sends the initial object of $\mathsf{P}\text{-}\mathbf{Alg}$ to the initial object of $\mathbf{OWC}$. Now, as we noted in Section 3·3, the functor $U : \mathbf{NGOpd} \to \mathbf{GOpd}$ has a left adjoint; and an application of the adjoint lifting theorem [**11**, theorem 2] shows that $\tilde{U}$ also has a left adjoint. Note that here we need the fact that $\mathsf{P}\text{-}\mathbf{Alg}$ is again describable in terms of essentially-algebraic structure, and so has coequalisers. Furthermore, $\tilde{U}$ is fully faithful, because U is and (3·3) is a pullback; and so we may identify $\mathsf{P}\text{-}\mathbf{Alg}$ with a reflective subcategory of $\mathbf{OWC}$. Thus we will be done if we can show that the initial object (L, λ) of $\mathbf{OWC}$ lies in this reflective subcategory; in other words, if we can show that L is normalised. But this is known to be the case: see [**5**], for example.

Remark 11. Note that the restriction to normalised globular operads is vital for the above proof to go through. Indeed, were we to take the universal cofibrant replacement for $\mathcal{T}$ in the category $\mathbf{GOpd}$ rather than $\mathbf{NGOpd}$, then we would no longer obtain Leinster's operad L. By following the steps of the above proof, we find that what we obtain instead is the initial operad-with-augmented-contraction, where an *augmented contraction* on a collection $C \in \mathbf{GColl}$ is given by a contraction on C together with a chosen element of the set $C(\star)$ of 0-dimensional operations. The initial operad-with-augmented-contraction is no longer normalised; and in fact, it is not hard to see that its monoid of 0-dimensional operations is the free monoid on one generator.

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