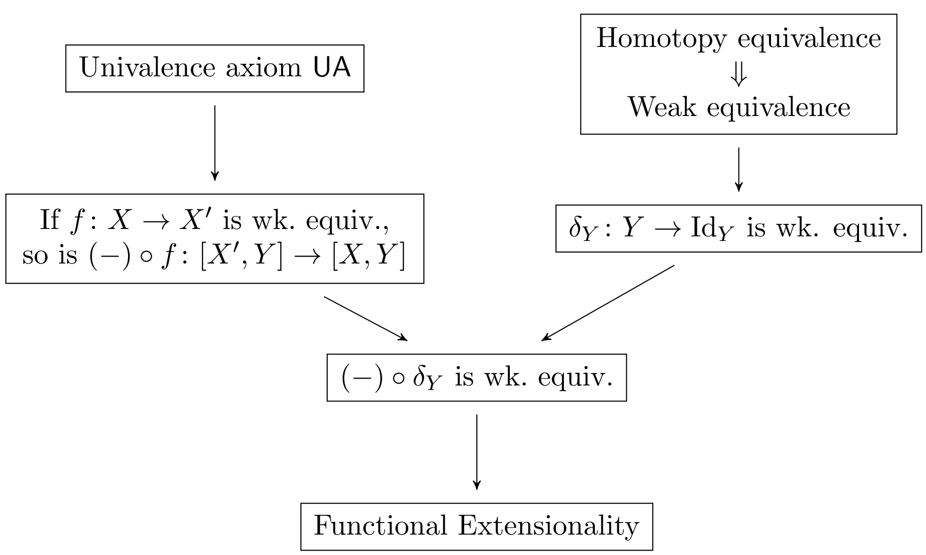
THE UNIVALENCE AXIOM AND FUNCTIONAL EXTENSIONALITY

TALK BY NICOLA GAMBINO; NOTES BY C. KAPULKIN, P. LEF. LUMSDAINE

These notes were taken and LaTeX'd by Chris Kapulkin and Peter LeFanu Lumsdaine, from Nicola Gambino's lecture in the Oberwolfach Mini-Workshop on the Homotopy Interpretation of Constructive Type Theory.

We present Vladimir Voevodsky's proof that the Univalence Axiom implies Functional Extensionality. The original proof was written in Coq code; here we present it in 'standard mathematical prose'.

We will proceed as follows. First, we introduce the notions of *weak equivalence* and *homotopy equivalence* of types, and show that these are equivalent. Since the diagonal map $\delta_X \colon X \to \mathrm{Id}(X)$ from a type to its total path space is a homotopy equivalence, it is hence also a weak equivalence. Next, we state the *Univalence Axiom* (UA), and show it implies that the map of function spaces given by precomposition with any weak equivalence is also a weak equivalence. Hence precomposition with δ_X is a weak equivalence. From this fact we derive *Functional Extensionality.*



We begin by fixing some notation and terminology. By the (propositional) η-rule for Π-types, we mean that any function is propositionally equal to its

η-expansion:

$$\frac{f : \Pi_{x:X} Y(x)}{\eta_f : \mathrm{Id}_{\Pi_{x:X} Y(x)}(\, f \,,\ (\lambda x\!:\!X)\, f x\,)}\ \Pi\text{-}\eta$$

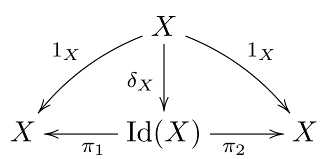
The formation and introduction rules for Id-types are taken to be:

$$\frac{X \ \mathsf{type} \qquad x, x' : X}{\mathrm{Id}_X(x, x') \ \mathsf{type}}\ \text{Id-FORM} \qquad \frac{x : X}{\mathrm{r}(x) : \mathrm{Id}_X(x, x')}\ \text{Id-INTRO}$$

By $\mathrm{Id}(X)$ we will denote the *total identity* type of a type X:

$$\mathrm{Id}(X) := \sum_{x, x' : X} \mathrm{Id}_X(x, x'),$$

whose elements are triples of the form $(x : X,\ x' : X,\ e : \mathrm{Id}_X(x, x'))$. This type comes equipped with the following maps:



where π_1, π_2 are the obvious projections, and δ_X maps $x\!:\!X$ to the triple $(x, x, \mathrm{r}(x))$.

We now introduce two classes of maps between types: weak equivalences and homotopy equivalences. For the former, we will need the notions of contractibility and a homotopy fiber of a map.

Definition 1. Let X be a type. We say that X is *contractible* if there is some $x_0\!:\!X$, such that for all $x\!:\!X$ we have an inhabitant $\alpha(x)$ of $\mathrm{Id}_X(x_0, x)$.

Definition 2. Given a map $f\colon X \to Y$ we define its *homotopy fiber* over $y\!:\!Y$ to be the type

$$\mathrm{hfiber}(f, y) := \sum_{x:X} \mathrm{Id}_Y(fx, y).$$

Definition 3. A map $f\colon X \to Y$ is a *weak equivalence* if for all $y\colon Y$, the homotopy fiber $\mathrm{hfiber}(f, y)$ is contractible.

Examples 4.

(1) Any identity map $1_X\colon X \to X$ is a weak equivalence.
(2) Suppose $(x : X)\ P(x)$ type and $e : \mathrm{Id}_X(x, x')$. Then the *transport* map $e^*\colon P(x) \to P(x')$ is a weak equivalence.

We denote the type of weak equivalences $f\colon X \to X'$ by $\mathrm{WEQ}(X, X')$.

Definition 5. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists some map $g\colon Y \to X$, inverse to f in that there are 'homotopies'

$$\eta : \prod_{x:X} \mathrm{Id}_X(x, gfx), \qquad \varepsilon : \prod_{y:Y} \mathrm{Id}_Y(fgy, y).$$

The following theorem gives a comparison between these classes of maps:

Theorem 6 (Grad Students' Lemma[1])**.** *A map $f\colon X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. The 'if' direction is routine. For the converse, use the 'type-theoretic axiom of choice'. ☐

The Grad Students' Lemma gives us an important corollary:

Corollary 7. *The diagonal map $\delta_Y\colon Y \to \mathrm{Id}(Y)$ is a homotopy equivalence (with inverse given by either projection), so it is a weak equivalence.*

We can now turn toward the Univalence Axiom. We begin by fixing a type universe U type, closed under the standard type constructors. Now, consider the identity types of U.

Definition 8. For any types $X, X' : \mathsf{U}$ and $e : \mathrm{Id}_{\mathsf{U}}(X, X')$, there is a weak equivalence $w_e\colon X \to X'$. In case $X = X'$, we define $w_{\mathrm{r}(X)} := 1_X$; this then extends inductively to all X, X', e.

Axiom 9 (Univalence)**.** *For all $X, X' : \mathsf{U}$, the canonical map*

$$w\colon \mathrm{Id}_{\mathsf{U}}(X, X') \to \mathrm{WEQ}(X, X')$$

is a weak equivalence.

As a consequence of the Univalence Axiom (UA) and the Grad Students Lemma, one obtains:

Fact 10. *We can derive rules asserting that every weak equivalence $f\colon X \to X'$ has a 'name' $\langle f\rangle : \mathrm{Id}_{\mathsf{U}}(X, X')$, and that this construction is inverse to w above:*

$$\frac{f\colon X \to X' \ \textit{w. e.}}{\langle f\rangle : \mathrm{Id}_{\mathsf{U}}(X, X')} \qquad \frac{e : \mathrm{Id}_{\mathsf{U}}(X, X')}{\eta_e : \mathrm{Id}_{\mathrm{Id}_{\mathsf{U}}(X,X')}(e, \langle w_e\rangle)} \qquad \frac{f\colon X \to X' \ \textit{w. e.}}{\varepsilon_f : \mathrm{Id}_{[X,X']}(w_{\langle f\rangle}, f)}$$

The next lemma will be the key step in proving Functional Extensionality from the Univalence Axiom.

Lemma 11. *If $X, X' : \mathsf{U}$, and $f\colon X \to X'$ is a weak equivalence, then for every type Y the map 'precomposition with f'*

$$(-) \circ f \ : \ [X', Y] \to [X, Y]$$

$$g\colon X' \to Y \quad \mapsto \quad g \circ f\colon X \to X' \to Y$$

is a weak equivalence.

Proof. Let $f\colon X \to X'$ be a weak equivalence. By Fact 10 we get $\langle f\rangle : \mathrm{Id}_{\mathsf{U}}(X, X')$. Fix any type Y, and consider the transport map $\langle f\rangle^*\colon [X', Y] \to [X, Y]$ obtained by applying Id-ELIM on $\langle f\rangle$.

[1]The name is due to Voevodsky.

Since $\langle f\rangle^*\colon [X',Y]\to[X,Y]$ is a weak equivalence (as a transport map), it is enough to show that

$$\prod_{u\colon X\to X'} \mathrm{Id}_{[X,Y]}(\langle f\rangle^*(u), u\circ f)$$

because then $(-)\circ f$ will be homotopic to a weak equivalence, and hence a weak equivalence.

However, by the second rule of Fact 10, it suffices to show that

$$\mathrm{Id}_{[X,Y]}(\langle f\rangle^*(u), u\circ w_{\langle f\rangle})$$

but because of the η-expansion we have

$$\mathrm{Id}_{[X,Y]}(e^*(u), u\circ w_e)$$

for any any $e : \mathrm{Id}_{\mathsf{U}}(X,X')$. □

Remark 12. Similarly, postcomposition with a weak equivalence gives a weak equivalence between the appropriate function spaces.

Out last lemma on the way to Functional Extensionality is a special case of it:

Lemma 13. *For any type $Y:\mathsf{U}$, the two projections $\pi_1,\pi_2\colon \mathrm{Id}(Y)\to Y$ are propositionally equal: that is, we have $\mathrm{Id}_{[\mathrm{Id}(Y),Y]}(\pi_1,\pi_2)$.*

Proof. Combining Lemma 11 with Corollary 7 we get that the map

$$(-)\circ\delta_Y\colon [\mathrm{Id}(Y),Y]\to[Y,Y]$$

is a weak equivalence. On the other hand we have

$$\mathrm{Id}_{[Y,Y]}(\pi_1\circ\delta_Y, \pi_2\circ\delta_Y)$$

so we must also have $\mathrm{Id}_{[\mathrm{Id}(Y),Y]}(\pi_1,\pi_2)$. □

Proof of Functional Extensionality. Let $f_1,f_2\colon X\to Y$ and

$$\phi : \prod_{x:X} \mathrm{Id}_Y(f_1x, f_2x).$$

Define $f\colon X\to\mathrm{Id}(Y)$ by $x\mapsto(f_1x,f_2x,\phi x)$. Now from Lemma 13 we have

$$\mathrm{Id}_{[X,Y]}(\pi_1\circ f, \pi_2\circ f),$$

completing the proof, as these composites are just η-expansions of f_1, f_2. □

As a final remark, we note two equivalents of Functional Extensionality:

Remark 14. The following are equivalent, for a given type X:

1. Functional Extensionality as used above: For all types Y, and $f,g\colon X\to Y$, if $\mathrm{Id}_Y(f(x),g(x))$ for all $x:X$, then $\mathrm{Id}_{[X,Y]}(f,g)$.
2. For all types Y, the canonical map $\mathrm{Id}([X,Y])\to[X,\mathrm{Id}(Y)]$ is a weak equivalence.
3. If for each $x:X$ we have a contractible type $P(x)$, then the product $\Pi_{x:X}P(x)$ is contractible.