# A Hamilton Jacobi Bellman Approach to Optimal Trade Execution *

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**Abstract**

The optimal trade execution problem is formulated in terms of a mean-variance tradeoff, as seen at the initial time. The mean-variance problem can be embedded in a Linear-Quadratic (LQ) optimal stochastic control problem, A semi-Lagrangian scheme is used to solve the resulting non-linear Hamilton Jacobi Bellman (HJB) PDE. This method is essentially independent of the form for the price impact functions. Provided a strong comparison property holds, we prove that the numerical scheme converges to the viscosity solution of the HJB PDE. Numerical examples are presented in terms of the efficient trading frontier and the trading strategy. The numerical results indicate that in some cases there are many different trading strategies which generate almost identical efficient frontiers.

**Keywords:** Optimal execution, mean-variance tradeoff, HJB equation, semi-Lagrangian discretization, viscosity solution

**AMS Classification** 65N06, 93C20

**Running Title:** An HJB Approach to Optimal Trading

## 1 Introduction

A large institutional investor, when selling a large block of shares, is faced with the following dilemma. If the investor trades rapidly, then the actual cash received from the sale will be less than anticipated, due to the market impact of the trades. Market impact can be minimized by breaking up a large trade into a number of smaller blocks. However, in this case, the investor is exposed to the risk of price depreciation during the trading horizon.

Recently, there has been considerable interest in algorithmic trading strategies. These are automated strategies for execution of trades with the objective of meeting pre-determined optimality criteria [14, 15].

In this work, we consider an idealized model for price impact. In the case of selling shares, the market price will decrease as a function of the trading rate, while at the same time following a stochastic process. The optimal control problem is then to liquidate the portfolio over some fixed time, and maximize the expected cash receipts while minimizing the variance of the outcome [9, 1, 2, 26, 16, 28].

An alternative approach is to pose this problem in terms of maximizing a power-law or exponential utility function [21, 32, 31]. Since a different objective function is used, the optimal strategies in [21, 32, 31] will, of course, be different from the strategy determined from the mean variance criteria. We will focus on the mean-variance approach in this work, due to its intuitive interpretation and popularity in industry.

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In [1], path-independent or static strategies are suggested. The optimal strategies are those which satisfy a mean-variance optimality condition, *recomputed at each trade time*. However, in [28], the authors acknowledge that this strategy cannot be optimal in terms of the mean-variance tradeoff as measured *at the initial time*. This subtle distinction is discussed in [26, 27, 8]. In [8], the strategy of maximizing the mean-variance objective at the initial time is termed the *pre-commitment policy*, i.e. once the initial strategy (as a function of the state variables) has been determined at the initial time, the trader commits to this policy, even if the optimal mean variance policy computed at a later time differs from the pre-commitment policy. This contrasts with the *time-consistent* policy, whereby the trader optimizes the mean-variance tradeoff at each instant in time, assuming optimal mean-variance strategies at each later instant. The advantages and disadvantages of these two different approaches are discussed in [8]. In this paper, we focus solely on the pre-commitment strategy, which is the optimal policy in terms of mean-variance as seen at the initial time.

A concrete example of the sense in which the pre-commitment strategy is optimal is the following. Suppose we are in an idealized world, where all our modelling assumptions (such as the form of the price impact functions, stochastic processes, and so on) are perfect. In this world, suppose we followed the pre-commitment strategy for many thousands of different trades. We then measure the standard deviation and expected gain (relative to the initial pre-trade state) averaged over the thousands of trades. Any other trading strategy (including the time-consistent strategy) would never produce a larger expected gain for a given standard deviation compared to the pre-commitment strategy.

We formulate the optimal trading problem as an optimal stochastic control problem, where the objective is to maximize the mean-variance tradeoff as measured at the initial time. The mean variance objective function can be converted to linear-quadratic (LQ) objective function using a Lagrange multiplier method [24, 10, 34, 4, 20]. Standard dynamic programming can then be used to derive a Hamilton-Jacobi-Bellman (HJB) PDE. Note that previously this method has been used mainly as a tool for obtaining analytic solutions to multi-period mean-variance investment problems. Analytic solutions are, of course, not available for many problems.

In this work, we the formulate the optimal trading problem in terms of the equivalent LQ formulation. We then use a numerical method to solve the resulting HJB equation for the optimal strategy. Our main contributions in this paper are

- We formulate the numerical problem so that a single solve of the nonlinear HJB problem, and a single solve of a related linear PDE, generates the entire efficient trading frontier.
- We develop a semi-Lagrangian scheme for solution of the HJB PDE and prove that this method is monotone, consistent and stable, hence converges to the viscosity solution of the HJB equation [7, 5] assuming that the HJB equation satisfies a strong comparison principle.
- We assume geometric Brownian motion for the stochastic process of the underlying asset, and a specific form for the price impact functions. However, our numerical method is essentially independent of any particular form for the price impact functions, and can be easily generalized to other stochastic processes (e.g. jump diffusion, regime switching). The technique is also amenable to implementation on multi-processor architectures.
- The trading problem is originally three dimensional. However, in some cases, the HJB PDE can be reduced to two dimensions using a similarity reduction. Our numerical formulation can be used for either the full three dimensional case, or for cases when the similarity reduction is valid, with minor modification.
- The numerical results indicate that there are some cases there are many different trading strategies which generate almost the same efficient frontier.

# 2 Optimal Execution

Let

$$\begin{aligned} S &= \text{Price of the underlying risky asset} \\ \alpha &= \text{Number of shares of underlying asset} \\ B &= \text{Risk free bank account .} \end{aligned} \tag{2.1}$$

At any time $t \in [0, T]$ an investor has a portfolio $\Pi$ given by

$$\Pi(t) \quad = \quad B + \alpha S \quad . \tag{2.2}$$

In order to handle both selling and buying cases symmetrically, we start off with $\alpha_I > 0$ shares if selling, and $\alpha_I < 0$ shares if buying. In other words, our objective is to liquidate a long position if selling, and to liquidate a short position if buying. More precisely

$$\begin{aligned} & t = 0 \rightarrow B = 0, S = S_0, \alpha = \alpha_I \\ & t = T \rightarrow B = B_L, S = S_T, \alpha = \alpha_T = 0 \\ & \qquad \alpha_I > 0 \text{ if selling} \\ & \qquad \alpha_I < 0 \text{ if buying} \end{aligned} \tag{2.3}$$

where $B_L$ is the cash which is generated by selling/buying in $[0, T)$, with a final liquidation/purchase at $t = T$ to ensure that the correct total number of shares are sold/bought. $B$ acts as a path dependent variable which keeps track of the total receipts obtained thus far from selling/buying the underlying asset $S$. Our objective will be to maximize $B_L$ and minimize the risk, as measured by the variance (or standard deviation) of $B_L$.

## 2.1 Problem Formulation: Overview

There are two popular formulations of the optimal trading problem. The impulse control formulation assumes that trades only take place at discrete points in time [21, 32]. However, this approach has the conceptual difficulty that the price impact of two discrete trades is independent of the time interval between trades. A better model would be based on impulse control (discrete trades) but include extra lag variables which would track the time interval between trades [29, 16]. However, this would be computationally expensive.

As a compromise, we can assume continuous trading at an instantaneous trading rate $v$ [28, 3]. This is unrealistic in the sense that real trading only takes place discretely. However, we can make the temporary price impact a function of the trade velocity, which introduces a simplified memory effect into the model, i.e. rapid trading has a larger temporary price impact than slower trading. We will use this model in the following.

## 2.2 Problem Formulation: Details

Let the trading rate $v$ be

$$v = \frac{d\alpha}{dt}, \tag{2.4}$$

where $\alpha$ is the number of shares in the portfolio (2.2).

For definiteness, we will suppose that $S$ follows geometric Brownian Motion (GBM), with a modification due to the permanent price impact of trading at rate $v$

$$\begin{aligned} dS &= (\eta + g(v))Sdt + \sigma S dZ \\ & \quad \eta \text{ is the drift rate of } S \\ & \quad g(v) \text{ is the permanent price impact} \\ & \quad \sigma \text{ is the volatility} \\ & \quad dZ \text{ is the increment of a Wiener process} \quad . \end{aligned} \tag{2.5}$$

We use the following form for the permanent price impact

$$\begin{aligned} g(v) &= \kappa_p v \\ & \quad \kappa_p \text{ is the permanent price impact factor} \quad . \end{aligned} \tag{2.6}$$

We take $\kappa_p$ to be a constant. Suppose $\eta = 0$, $\sigma = 0$ in equation (2.5). If $X = \log S$, then from equations (2.5-2.6) we have

$$X(t) - X(0) = \kappa_p \int_0^t v(u) \, du \tag{2.7}$$

which means that $X(t) = X(0)$ if a round-trip trade ($\int_0^t v(u) \, du = 0$) is executed. This form of permanent price impact eliminates round-trip arbitrage opportunities [22, 3].

The bank account $B$ is assumed to follow

$$\frac{dB}{dt} = rB - vS \, f(v) \tag{2.8}$$

$$\begin{aligned} & r \text{ is the risk-free return} \\ & f(v) \text{ is the temporary price impact and transaction cost function} \quad . \end{aligned} \tag{2.9}$$

The term $vS \, f(v)$ represents the rate of cash expended to purchase shares at price $S \, f(v)$ at a rate $v$. The temporary price impact and transaction cost function $f(v)$ is assumed to be

$$\begin{aligned} f(v) &= [1 + \kappa_s \operatorname{sgn}(v)] \exp[\kappa_t \operatorname{sgn}(v)|v|^\beta] \\ & \quad \kappa_s \text{ is the bid-ask spread parameter} \\ & \quad \kappa_t \text{ is the temporary price impact factor} \\ & \quad \beta \text{ is the price impact exponent} \quad . \end{aligned} \tag{2.10}$$

We shall refer to $f(v)$ in the following as the temporary price impact function, although strictly speaking, we also include a transaction cost term as well. For various studies which suggest the form (2.10) see [25, 30, 3].

Given the state variables $(S, B, \alpha)$ the instant before the end of trading $t = T^-$, then we have one final trade (if necessary) so that the number of shares owned at $t = T$ is $\alpha_T = 0$, as in equation (2.3). The liquidation value after this final trade $B_L = \Phi^L(S, \alpha, B, \alpha_T)$ is determined from a discrete form of equation (2.8) i.e.

$$B_L = \Phi^L(S, B, \alpha, \alpha_T) = B - v_T(\Delta t)_T S f(v_T) \ , \tag{2.11}$$

where $v_T$ is given from

$$v_T = \frac{\alpha_T - \alpha}{(\Delta t)_T} = \frac{-\alpha}{(\Delta t)_T} \tag{2.12}$$

where we can specify that the liquidation interval is very short, e.g. $(\Delta t)_T = 10^{-5}$ years. Note that effectively the liquidation value (2.11) penalizes the trader for not hitting the target $\alpha = \alpha_T$ at the end of trading. The optimal strategy will attempt to avoid this state (where $\alpha \neq \alpha_T$), hence the results are insensitive to $(\Delta t)_T$ if this value is selected sufficiently small. In the case of selling, $B_L$ will be a positive quantity obtained by selling $\alpha_I$ shares. In the case of buying, $B_L$ will be negative, indicating a cash outflow to liquidate a short position of $\alpha_I$ shares (i.e. buying $|\alpha_I|$ shares).

## 2.3 The Optimal Strategy

Let $v(S, B, \alpha, t)$ be a specified trading strategy. Let $E^{t=0}_{v(\cdot)}[B_L]$ be the expected gain from this strategy. Define the variance of the gain for this strategy as

$$Var^{t=0}_{v(\cdot)}[B_L] \quad = \quad E^{t=0}_{v(\cdot)}[(B_L)^2] - (E^{t=0}_{v(\cdot)}[B_L])^2 \ . \tag{2.13}$$

The control problem is then to determine the optimal strategy $v^*(S, B, \alpha, t)$ such that $E^{t=0}_{v^*(\cdot)}[B_L] = d$, while minimizing the risk as measured by the variance. More formally, we seek the strategy $v^*(\cdot)$ which solves the problem

$$\begin{aligned} &\min Var^{t=0}_{v(\cdot)}[B_L] = E^{t=0}_{v(\cdot)}[(B_L)^2] - d^2 \\ &\text{subject to } \begin{cases} E^{t=0}_{v(\cdot)}[B_L] = d \\ v(\cdot) \in Z \end{cases} , \end{aligned} \tag{2.14}$$

where $Z$ is the set of admissible controls. We emphasize here that the expectation and variance are *as seen at* $t = 0$.

Problem (2.14) determines the best strategy given a specified $E^{t=0}_{v(\cdot)}[B_L] = d$. Varying the expected value $d$ traces out a curve in the expected value, standard deviation plane. This curve is known as an *efficient frontier*. Each point on the curve represents a trading strategy which is optimal in the sense that there is no other strategy which gives rise to a smaller risk for the given expected value of the trading gain. Consequently, any rational trader will only choose strategies which correspond to points on the efficient frontier. Different traders will, however, choose different points on the efficient frontier, which will depend on their risk preferences.

## 2.4 Objective Function: Efficient Frontier

Problem (2.14) is a convex optimization problem, and hence has a unique solution. We can eliminate the constraint in problem (2.14) by using a Lagrange multiplier [24, 10, 34, 4, 20], which we denote by $\gamma$. Problem (2.14) can then be posed as [11]

$$\max_{\gamma} \min_{v(\cdot)\in Z} E^{t=0}_{v(\cdot)}\left[(B_L)^2 - d^2 - \gamma(E^{t=0}_{v(\cdot)}[B_L] - d)\right] \ . \tag{2.15}$$

For fixed $\gamma, d$, this is equivalent to finding the control $v(\cdot)$ which solves

$$\min_{v(\cdot)\in Z} E^{t=0}_{v(\cdot)}[(B_L - \frac{\gamma}{2})^2] \ . \tag{2.16}$$

Note that if for some fixed $\gamma$, $v^*(\cdot)$ is the optimal control of problem (2.16), then $v^*(\cdot)$ is also the optimal control of problem (2.14) with $d = E^{t=0}_{v^*}[B_L]$ [24, 10], where the notation $E^{t=0}_{v^*}[\cdot]$ refers to the expected value given the strategy $v^*(\cdot)$. Conversely, if there exists a solution to problem (2.14), with $E^{t=0}_{v^*}[B_L] = d$, then there exists a $\gamma$ which solves problem (2.16) with control $v^*(\cdot)$. We can now restrict attention to solving problem (2.16).

For a given $\gamma$, finding the control $v^*(\cdot)$ which minimizes equation (2.16) gives us a single pair $(E_{v^*}[B_L], Var_{v^*}[B_L])$ on the variance minimizing efficient frontier. Varying $\gamma$ allows us to trace out the entire frontier.

**Remark 2.1** (Efficient Frontier)**.** *The efficient frontier, as normally defined, is a portion of the variance minimizing frontier [10]. That is, given a point* $(E_{v^*}[B_L], \sqrt{Var_{v^*}[B_L]})$ *on the efficient frontier, corresponding to control* $v^*(\cdot)$*, then there exists no other control* $\bar{v}^*(\cdot)$ *such that* $Var_{\bar{v}^*}[B_L] = Var_{v^*}[B_L]$ *with* $E_{\bar{v}^*}[B_L] > E_{v^*}[B_L]$*. Hence the points on the efficient frontier are Pareto optimal [35]. From a computational perspective, once a set of points on the variance minimizing frontier are determined, then the efficient frontier can be be constructed by a simple sorting operation.*

We will assume that the set of admissible controls is given by

$$\begin{aligned} Z &\in [v_{\min}, v_{\max}] \\ & \quad v_{\min} \le 0 \le v_{\max} \end{aligned} \tag{2.17}$$

If only selling is permitted, then, for example,

$$\begin{aligned} v_{\min} &< 0 \\ v_{max} &= 0 \ . \end{aligned} \tag{2.18}$$

$v_{\min}, v_{\max}$ are assumed to be bounded in the following.

Bearing in mind that we are going to solve problem (2.16) by solving the corresponding Hamilton-Jacobi-Bellman control PDE, we would like to avoid having to do many PDE solves. Define (assuming $\gamma = const.$)

$$\mathcal{B}(t) = B(t) - \frac{\gamma e^{-r(T-t)}}{2} \ . \tag{2.19}$$

Then let

$$\begin{aligned} \mathcal{B}_L &= \Phi_L(S, \mathcal{B}(t = T^-), \alpha, \alpha_T) \\ &= \Phi_L(S, B(t = T^-), \alpha, \alpha_T) - \frac{\gamma}{2} \\ &= B_L - \frac{\gamma}{2} \ , \end{aligned} \tag{2.20}$$

so that problem (2.16) becomes, in terms of $\mathcal{B}_L = B_L - \gamma/2$

$$\min_{v(\cdot) \in Z} E^{t=0}[\mathcal{B}_L^2] \ . \tag{2.21}$$

Note (from equations (2.8), (2.19)) that

$$\frac{d\mathcal{B}}{dt} = r\mathcal{B} - vS \ f(v) \tag{2.22}$$

which has the same form as equation (2.8).

However, we now have the $\gamma$ dependence appearing at $t = 0$. Recall from equation (2.3) that $B(t = 0) = 0$, then

$$t = 0 \rightarrow \mathcal{B} = \frac{-\gamma e^{-rT}}{2}, S = S_0, \alpha = \alpha_I \ . \tag{2.23}$$

This is very convenient, in the PDE context. We simply determine the numerical solution for problem (2.21), which is independent of $\gamma$. We can then determine the solution for different discrete values of $\gamma$ by examining the solution for different discrete values of $\mathcal{B}(t = 0)$. Since we normally solve the PDE for a range of discrete values of $\mathcal{B}$, we can solve problem (2.21) once, and use this result to construct the entire variance minimizing efficient frontier.

# 3 HJB Formulation: Overview

## 3.1 Determination of Optimal Control

Let $V = V(S, \mathcal{B}, \alpha, \tau = T - t) = E^t[\mathcal{B}_L^2]$ and denote

$$\mathcal{L}V \equiv \frac{\sigma^2 S^2}{2} V_{SS} + \eta S V_S \ . \tag{3.1}$$

Assuming process (2.5), and equations (2.4), (2.22), then following standard arguments [17], the solution to problem (2.21) is given from the solution to

$$\begin{aligned} V_\tau &= \mathcal{L}V + r\mathcal{B}V_\mathcal{B} + \min_{v\in Z}\Big[-vSf(v)V_\mathcal{B} + vV_\alpha + g(v)SV_S\Big] \\ & \quad Z = [v_{min}, v_{max}] \end{aligned} \tag{3.2}$$

with the initial condition (at $\tau = 0$ or $t = T$)

$$V(S,\mathcal{B},\alpha,\tau=0) = \mathcal{B}_L^2 \ , \tag{3.3}$$

where $\mathcal{B}_L$ is given from equation (2.20). Solution of this problem determines an optimal control $v^*(S,\mathcal{B},\alpha,\tau)$ at each point $(S,\mathcal{B},\alpha,\tau)$. We can use equation (2.19) to determine the control in terms of the variables $(S,B,\alpha,\tau)$.

## 3.2 Determination of Expected Value

We need to determine $E^{t=0}_{v^*}[\mathcal{B}_L]$ in order to determine the pair $(E^{t=0}_{v^*}[B_L],(E^{t=0}_{v^*}[B_L^2])$ which generates a point on the variance minimizing efficient frontier for a given $\gamma$.

Let $U = U(S,\mathcal{B},\alpha,\tau = T-t) = E^t_{v^*}[\mathcal{B}_L]$. The operator $\mathcal{L}U$ is defined as in equation (3.1). Let $v^*(S,\mathcal{B},\alpha,\tau)$ be the optimal control from problem (3.2). Once again, assuming process (2.5), then $U$ satisfies

$$U_\tau = \mathcal{L}U + r\mathcal{B}U_\mathcal{B} - v^*Sf(v^*)U_\mathcal{B} + v^*U_\alpha + g(v^*)SU_S \tag{3.4}$$

with the initial condition

$$U(S,\mathcal{B},\alpha,\tau=0) = \mathcal{B}_L \tag{3.5}$$

where $\mathcal{B}_L$ is given from equation (2.20). Since the most costly part of the solution of equation (3.2) is the determination of the optimal control $v^*$, solution of equation (3.4) is very inexpensive, since $v^*$ is known.

## 3.3 Construction of the Efficient Frontier

Once we have solved problems (3.2) and (3.4) we can now construct the efficient frontier.

We examine the solution values at $\tau = T (t = 0)$ for the initial values of $(S,\alpha)$ of interest. Define

$$\begin{aligned} V_0(\mathcal{B}) &= V(S=S_0,\mathcal{B},\alpha=\alpha_I,\tau=T) \\ U_0(\mathcal{B}) &= U(S=S_0,\mathcal{B},\alpha=\alpha_I,\tau=T) \ . \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} V_0(\mathcal{B}) &= E^{t=0}_{v^*}[\mathcal{B}_L^2] \\ U_0(\mathcal{B}) &= E^{t=0}_{v^*}[\mathcal{B}_L] \ . \end{aligned} \tag{3.7}$$

From equation (2.23), a value of $\mathcal{B}$ at $t = 0$ or $\tau = T$ corresponds to the value of $\gamma$ given by

$$\gamma = -2e^{rT}\mathcal{B} \ . \tag{3.8}$$

Note that $E^{t=0}_{v^*}[y(\mathcal{B})]$ for known $v^*$ is given from the solution to linear PDE (3.4), with initial condition $y(\mathcal{B})$, so that $E^{t=0}_{v^*}[const.] = const.$ Recall $\mathcal{B}_L = B_L - \gamma/2$, so that from equations (3.7) we have

$$\begin{aligned} V_0(\mathcal{B}) &= E^{t=0}_{v^*}[B_L^2] - \gamma E^{t=0}_{v^*}[B_L] + \frac{\gamma^2}{4} \\ U_0(\mathcal{B}) &= E^{t=0}_{v^*}[B_L] - \frac{\gamma}{2} \ , \end{aligned} \tag{3.9}$$

with $\gamma = \gamma(\mathcal{B})$ from equation (3.8).

Consequently, for given $\mathcal{B}$, $\gamma$ is given from equation (3.8), then $E^{t=0}_{v^*}[B_L^2]$ and $E^{t=0}_{v^*}[B_L]$ are obtained from equations (3.9). By examining the solution for different values of $\mathcal{B}$, we trace out the entire variance minimizing efficient frontier.

**Remark 3.1** (Generation of the efficient points)**.** *As discussed in Remark 2.1, the points on the efficient frontier are, in general, a subset of the points on the variance minimizing frontier. Given a set of points on the variance minimizing frontier, the points are sorted in order of increasing expected value. Then these points are traversed in order from the highest expected value to the lowest expected value. Any points which have a higher variance compared to a previously examined point are rejected.*

### 3.4 Similarity Reduction

For price impact functions of the form (2.6) and (2.10), payoffs (3.3) and (3.5), and assuming geometric Brownian Motion (2.5) then

$$\begin{aligned} V(\xi S, \xi\mathcal{B}, \alpha, \tau) &= \xi^2 V(S, \mathcal{B}, \alpha, \tau) \\ U(\xi S, \xi\mathcal{B}, \alpha, \tau) &= \xi U(S, \mathcal{B}, \alpha, \tau) \ . \end{aligned} \tag{3.10}$$

Consequently,

$$V(S, \mathcal{B}, \alpha, \tau) = \left(\frac{\mathcal{B}}{\mathcal{B}^*}\right)^2 V(\frac{\mathcal{B}^* S}{\mathcal{B}}, \mathcal{B}^*, \alpha, \tau) \tag{3.11}$$

$$U(S, \mathcal{B}, \alpha, \tau) = \left(\frac{\mathcal{B}}{\mathcal{B}^*}\right) U(\frac{\mathcal{B}^* S}{\mathcal{B}}, \mathcal{B}^*, \alpha, \tau) \ . \tag{3.12}$$

and hence we need only solve for two fixed values of $\mathcal{B}^*$, (one positive and one negative) and we can reduce the numerical computation to (essentially) a two dimensional problem (see Section 5.1).

## 4 HJB Formulation: Details

Consequently, the problem of determining the efficient frontier reduces to solving equations (3.2) and (3.4).

### 4.1 Determination of the Optimal Control

Equation (3.2) is

$$V_\tau = \mathcal{L}V + r\mathcal{B}V_{\mathcal{B}} + \min_{v \in Z}\left[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\right] \ . \tag{4.1}$$

The domain of equation (4.1) is

$$(S, \mathcal{B}, \alpha, \tau) \in [0, \infty] \times [-\infty, +\infty] \times [\alpha_{min}, \alpha_{\max}] \times [0, T] \ , \tag{4.2}$$

where, for example $\alpha_{min} = \min(0, \alpha_I)$, $\alpha_{\max} = \max(\alpha_I, 0)$ if we only allow monotonic buying/selling. We also typically normalize quantities so that $|\alpha_I| = 1$. For numerical purposes, we localize the domain (4.2) to

$$(S, \mathcal{B}, \alpha, \tau) \in [0, S_{\max}] \times [B_{\min}, B_{\max}] \times [\alpha_{min}, \alpha_{\max}] \times [0, T] \ . \tag{4.3}$$

At $\alpha = \alpha_{min}, \alpha_{max}$, we do not allow buying/selling which would cause $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$, so that

$$\begin{aligned} V_\tau &= \mathcal{L}V + r\mathcal{B}V_{\mathcal{B}} + \min_{v \in Z^-}\left[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\right] \\ & \alpha = \alpha_{max} \ ; \ Z^- = [v_{min}, 0] \end{aligned} \tag{4.4}$$

$$\begin{aligned} V_\tau &= \mathcal{L}V + r\mathcal{B}V_{\mathcal{B}} + \min_{v \in Z^+}\left[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\right] \\ & \alpha = \alpha_{min} \ ; \ Z^+ = [0, v_{\max}] \ . \end{aligned} \tag{4.5}$$

At $\mathcal{B} = \mathcal{B}_{\min}, \mathcal{B}_{\max}$, we can assume that equation (3.11) holds. In which case, we can replace $V_\mathcal{B}$ in equation (4.1) by

$$V_\mathcal{B} = \frac{2}{\mathcal{B}} V - \frac{S}{\mathcal{B}} V_S \quad ; \quad \mathcal{B} = \mathcal{B}_{\min}, \mathcal{B}_{\max} \quad . \tag{4.6}$$

In general, this would be an approximation. However, in our case, equation (3.11) holds exactly. In fact, we will not need to consider boundary conditions at $\mathcal{B}_{\min}, \mathcal{B}_{\max}$ since we will use equation (3.11) to effectively eliminate the $\mathcal{B}$ variable. We include equation (4.6) for generality.

The initial condition is

$$V(S, \mathcal{B}, \alpha, 0) = (\mathcal{B}_L)^2 \ . \tag{4.7}$$

At $S = 0$, no boundary condition is required for equation (4.1), we simply solve equation (4.1) with $\mathcal{L}V = 0$. At $S \to \infty$, consider the cases of buying and selling separately. In the case of selling, we would normally have $0 \le \alpha \le \alpha_I$, so that $\alpha f(v) \to 0$ if $(\Delta t)_T \to 0$ in equation (2.11). Hence $\mathcal{B}_L \simeq \mathcal{B}$ which is independent of $S$. For $\tau > 0$, the optimal strategy for $S$ large will attempt to find the solution which minimizes $\mathcal{B}^2$, so the value will also be independent of $S$ as $S \to \infty$.

In the case of buying, $(S \to \infty)$

$$\mathcal{B}_L^2 \simeq \alpha^2 (S f(v_T))^2 \ . \tag{4.8}$$

In this case, the payoff condition essentially penalizes the trader for not meeting the target value of $\alpha_T = 0$ the instant before trading ends when $S$ is large. The optimal strategy would therefore be to make sure $\alpha \simeq 0$ at $t \to T$. Hence the optimal control at $\tau > 0$ when $S \to \infty$ should tend to force $\alpha = 0$. In other words, from equations (2.11), (4.8), $V(S_{\max}, \mathcal{B}, \alpha, \tau > 0) \simeq V(S_{\max}, \mathcal{B}, \alpha_T, \tau) \simeq \mathcal{B}^2$, which is independent of $S$. Hence, in both cases, we make the *ansatz* that

$$V_{SS}, V_S \to 0 \quad ; \quad S = S_{\max} \ , \tag{4.9}$$

so that equation (4.1) becomes

$$V_\tau = r\mathcal{B} V_\mathcal{B} + \min_{v \in Z} \Big[ -vSf(v) V_\mathcal{B} + v V_\alpha \Big] \quad ; \quad S = S_{\max} \quad . \tag{4.10}$$

Equation (4.10) is clearly an approximation, but has the advantage that it is very easy to implement. We shall carry out various numerical tests with different values of $S_{\max}$ to show that the error in this approximation can be made small in regions of interest.

## 4.2 Determination of the Expected Value

Given the optimal trading strategy $v^* = v^*(S, \mathcal{B}, \alpha, \tau)$ determined from equation (4.1), the expected value $U = E_{v^*}^{t=0}[\mathcal{B}_L]$ is given from equation (3.4)

$$U_\tau = \mathcal{L}U + r\mathcal{B} U_\mathcal{B} - v^* S f(v^*) V_\mathcal{B} + v^* V_\alpha + g(v^*) S V_S \ . \tag{4.11}$$

At $S = 0$ we simply solve equation (4.11). From equation (4.4), at $\alpha = \alpha_{\max}$, we must have $v^*(S, \mathcal{B}, \alpha_{\max}, \tau) \le 0$ hence no boundary condition is required at $\alpha = \alpha_{\max}$. Similarly, at $\alpha = \alpha_{\min}$, $v^*(S, \mathcal{B}, \alpha_{\min}, \tau) \ge 0$, and no boundary condition is required at $\alpha = \alpha_{\min}$. The boundary conditions at $\mathcal{B} = \mathcal{B}_{\min}, \mathcal{B}_{\max}$ can be eliminated using equation (3.12)

$$U_B = \frac{1}{\mathcal{B}} U - \frac{S}{\mathcal{B}} U_S \quad ; \quad \mathcal{B} = \mathcal{B}_{\min}, \mathcal{B}_{\max} \quad . \tag{4.12}$$

However, in this paper, the similarity reduction (3.12) is exact, hence we can eliminate the $\mathcal{B}$ variable, and thus no boundary condition at $\{\mathcal{B}_{\min}, \mathcal{B}_{\max}\}$ is required.

Following similar arguments as used in deriving equation (4.10), we assume $U_S, U_{SS} \to 0$ as $S \to S_{\max}$, hence equation (4.11) becomes

$$U_\tau \quad = \quad r\mathcal{B}U_\mathcal{B} - v^* S f(v^*) V_\mathcal{B} + v^* V_\alpha \quad ; \quad S = S_{\max} \quad . \tag{4.13}$$

The payoff condition is

$$U(S, \mathcal{B}, \alpha, 0) = \mathcal{B}_L \quad . \tag{4.14}$$

# 5 Discretization: An Informal Approach

We first provide an informal discretization of equation (4.1) using a semi-Lagrangian approach. We prove that this is a consistent discretization in Section A.3. Equation (4.11) is discretized in a similar fashion. The reader is referred to the references in [12] for more details concerning semi-Lagrangian methods for HJB equations.

Along the trajectory $S = S(\tau), \mathcal{B} = \mathcal{B}(\tau), \alpha = \alpha(\tau)$ defined by

$$\begin{aligned} \frac{dS}{d\tau} &= -g(v)S \\ \frac{d\mathcal{B}}{d\tau} &= -\left(r\mathcal{B} - vSf(v)\right) \\ \frac{d\alpha}{d\tau} &= -v \quad , \end{aligned} \tag{5.1}$$

equation (4.1) can be written as

$$\max_{v \in Z} \frac{DV}{D\tau} \quad = \quad \mathcal{L}V \quad , \tag{5.2}$$

where the Lagrangian derivative $DV/D\tau$ is given by

$$\frac{DV}{D\tau} \quad = \quad V_\tau - V_S g(v) S - V_\mathcal{B}\left(r\mathcal{B} - vSf(v)\right) - V_\alpha v \quad . \tag{5.3}$$

The Lagrangian derivative is the rate of change of $V$ along the trajectory (5.1).

Define a set of nodes $[S_0, S_1, ..., S_{i_{max}}]$, $[\mathcal{B}_0, \mathcal{B}_1, ..., \mathcal{B}_{j_{max}}]$, $[\alpha_0, \alpha_1, ..., \alpha_{k_{max}}]$, and discrete times $\tau^n = n\Delta\tau$. Let $V(S_i, \mathcal{B}_j, \alpha_k, \tau^n)$ denote the exact solution to equation (4.1) at point $(S_i, \mathcal{B}_j, \alpha_k, \tau^n)$. Let $V^n_{i,j,k}$ denote the discrete approximation to the exact solution $V(S_i, \mathcal{B}_j, \alpha_k, \tau^n)$.

We use standard finite difference methods [13] to discretize the operator $\mathcal{L}V$ as given in (3.1). Let $(\mathcal{L}_h V)^n_{i,j,k}$ denote the discrete value of the differential operator (3.1) at node $(S_i, \mathcal{B}_j, \alpha_k, \tau^n)$. The operator (3.1) can be discretized using central, forward, or backward differencing in the $S$ direction to give

$$(\mathcal{L}_h V)^n_{i,j,k} = a_i V^n_{i-1,j,k} + b_i V^n_{i+1,j,k} - (a_i + b_i) V^n_{i,j,k} \ , \quad i < i_{\max}, \tag{5.4}$$

where $a_i$ and $b_i$ are determined using an algorithm in [13]. The algorithm guarantees $a_i$ and $b_i$ satisfy the following positive coefficient condition:

$$a_i \geq 0 \quad ; \quad b_i \geq 0 \ , \quad i = 0, \ldots, i_{\max}. \tag{5.5}$$

The boundary conditions will be taken into account by setting

$$\begin{aligned} a_0 &= a_{i_{max}} = 0 \\ b_0 &= b_{i_{max}} = 0 \quad . \end{aligned} \tag{5.6}$$

Define the vector $V^n_{j,k} = [V^n_{0,j,k}, ..., V^n_{i_{\max},j,k}]^t$, then $\mathcal{L}_h$ is an $i_{\max} + 1 \times i_{\max} + 1$ matrix such that $(\mathcal{L}_h V^n_{j,k})_i$ is given by equation (5.4).

Let $v^n_{i,j,k}$ denote the approximate value of the control variable $v$ at mesh node $(S_i, \mathcal{B}_j, \alpha_k, \tau^n)$. Then we approximate $DV/D\tau$ at $(S_i, \mathcal{B}_j, \alpha_k, \tau^{n+1})$ by the following

$$\left(\frac{DV}{D\tau}\right)^{n+1}_{i,j,k} \simeq \frac{1}{\Delta\tau}(V^{n+1}_{i,j,k} - V^n_{\hat{i},\hat{j},\hat{k}}) \tag{5.7}$$

where $V^n_{\hat{i},\hat{j},\hat{k}}$ is an approximation of $V(S^n_{\hat{i}}, \mathcal{B}^n_{\hat{j}}, \alpha^n_{\hat{k}}, \tau^n)$ obtained by linear interpolation of the discrete values $V^n_{i,j,k}$, with $(S^n_{\hat{i}}, \mathcal{B}^n_{\hat{j}}, \alpha^n_{\hat{k}})$ given by solving equations (5.1) backwards in time, from $\tau^{n+1}$ to $\tau^n$, for fixed $v^{n+1}_{i,j,k}$ to give (noting that $g(v^{n+1}_{i_{\max},j,k}) = 0$ from equation (4.10))

$$\begin{aligned}
S^n_{\hat{i}} &= S_i \exp[g(v^{n+1}_{i,j,k})\Delta\tau] \;\; ; \;\; i < i_{imax} \\
&= S_i \;\; ; \;\; i = i_{imax} \\
\mathcal{B}^n_{\hat{j}} &= \mathcal{B}_j \exp[r\Delta\tau] - v^{n+1}_{i,j,k} S_i f(v^{n+1}_{i,j,k}) \left( \frac{e^{r\Delta\tau} - e^{g(v^{n+1}_{i,j,k})\Delta\tau}}{r - g(v^{n+1}_{i,j,k})} \right) \\
\alpha^n_{\hat{k}} &= \alpha_k + v^{n+1}_{i,j,k}\Delta\tau \;\; .
\end{aligned} \tag{5.8}$$

Equation (5.8) is equivalent to $O((\Delta\tau)^2)$ to

$$\begin{aligned}
S^n_{\hat{i}} &= S_i + S_i g(v^{n+1}_{i,j,k})\Delta\tau + O(\Delta\tau)^2 \;\; ; \;\; i < i_{\max} \\
\mathcal{B}^n_{\hat{j}} &= \mathcal{B}_j + \big(r\mathcal{B}_j - v^{n+1}_{i,j,k} S_i f(v^{n+1}_{i,j,k})\big)\Delta\tau + O(\Delta\tau)^2 \\
\alpha^n_{\hat{k}} &= \alpha_k + v^{n+1}_{i,j,k}\Delta\tau \;\; .
\end{aligned} \tag{5.9}$$

For numerical purposes, we use equation (5.8) since this form ensures, for example, that $S^n_{\hat{i}} \geq 0$, regardless of timestep size. We will use the limiting form (5.9) when carrying out our consistency analysis.

All the information about the price impact function is embedded in equation (5.8). This means that the form of the price impact functions can be easily altered, with minimal changes to an implementation.

Let $Z^{n+1}_{i,j,k} \subseteq Z$ denote the set of possible values for $v^{n+1}_{i,j,k}$ such that $(S^n_{\hat{i}}, \mathcal{B}^n_{\hat{j}}, \alpha^n_{\hat{k}})$ remains inside the computational domain. In other words, $v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}$ ensures that

$$\begin{aligned}
&0 \leq S^n_{\hat{i}} \leq S_{i_{max}} \\
&\alpha_0 \leq \alpha^n_{\hat{k}} \leq \alpha_{k_{max}} \;\; .
\end{aligned} \tag{5.10}$$

Note that we do not impose any constraints to ensure $\mathcal{B}^n_{\hat{j}} \in [\mathcal{B}_{min}, \mathcal{B}_{max}]$. We will essentially eliminate the $\mathcal{B}$ variable using the similarity reduction (3.12).

We approximate the HJB PDE (4.1) and the boundary conditions (4.4-4.5), and (4.10) by

$$\begin{aligned}
V^{n+1}_{i,j,k} &= \min_{v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}} V^n_{\hat{i},\hat{j},\hat{k}} + \Delta\tau(\mathcal{L}_h V)^{n+1}_{i,j,k} \\
&\quad (v^*)^{n+1}_{i,j,k} \in \operatorname*{arg\,min}_{v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}} V^n_{\hat{i},\hat{j},\hat{k}} \;\; .
\end{aligned} \tag{5.11}$$

At $\tau^0 = 0$ we have the payoff condition (3.3)

$$V^0_{i,j,k} = ((\mathcal{B}_L)_{i,j,k})^2 \;\; . \tag{5.12}$$

Once the optimal control $(v^*)^{n+1}_{i,j,k} = v^*(S_i, \mathcal{B}_j, \alpha_k, \tau^{n+1})$ is determined from the solution to equation (5.11), then the solution to equation (4.13) is given by solving the linear PDE

$$U^{n+1}_{i,j,k} = \left\{ U^n_{\hat{i},\hat{j},\hat{k}} \right\}_{v = (v^*)^{n+1}_{i,j,k}} + \Delta\tau(\mathcal{L}_h U)^{n+1}_{i,j,k} \;\; , \tag{5.13}$$

with payoff condition

$$U^0_{i,j,k} = (\mathcal{B}_L)_{i,j,k} \;\; . \tag{5.14}$$

## 5.1 Discrete Similarity Reduction

If the similarity reduction (3.12) is valid (which is the case for the price impact functions, payoff and price process assumed in this work), we can reduce the number of nodes needed in the $\mathcal{B}$ direction to a finite number, independent of the mesh size.

Choose $\mathcal{B}^* > 0$, let $\mathcal{B}_j \in \mathcal{B}_{set} = \{-\mathcal{B}^*, +\mathcal{B}^*\}$, i.e. we have only two nodes in the discrete $\mathcal{B}$ grid. Further, let $\mathcal{B}_0 = -\mathcal{B}^*, \mathcal{B}_1 = +\mathcal{B}^*$. If $\mathcal{B}^n_{\hat{j}} > 0$ then we evaluate $V^n_{\hat{\imath},\hat{j},\hat{k}}, U^n_{\hat{\imath},\hat{j},\hat{k}}$ by

$$
\begin{aligned}
V^n_{\hat{\imath},\hat{j},\hat{k}} &= \left(\frac{\mathcal{B}^n_{\hat{j}}}{\mathcal{B}^*}\right)^2 V^n_{\hat{\imath}^*,1,\hat{k}} \\
U^n_{\hat{\imath},\hat{j},\hat{k}} &= \left(\frac{\mathcal{B}^n_{\hat{j}}}{\mathcal{B}^*}\right) U^n_{\hat{\imath}^*,1,\hat{k}} \\
& S_{\hat{\imath}^*} = \frac{\mathcal{B}^* S_{\hat{\imath}}}{\mathcal{B}^n_{\hat{j}}}
\end{aligned}
\tag{5.15}
$$

where $V^n_{\hat{\imath}^*,1,\hat{k}}$ refers to a linear interpolant of $V^n$ at the node $(S_{\hat{\imath}^*}, \mathcal{B}^*, \alpha_{\hat{k}})$.

If $\mathcal{B}^n_{\hat{j}} < 0$ then we evaluate $V^n_{\hat{\imath},\hat{j},\hat{k}}$ by

$$
\begin{aligned}
V^n_{\hat{\imath},\hat{j},\hat{k}} &= \left(\frac{\mathcal{B}^n_{\hat{j}}}{-\mathcal{B}^*}\right)^2 V^n_{\hat{\imath}^*,0,\hat{k}} \\
U^n_{\hat{\imath},\hat{j},\hat{k}} &= \left(\frac{\mathcal{B}^n_{\hat{j}}}{-\mathcal{B}^*}\right) U^n_{\hat{\imath}^*,0,\hat{k}} \\
& S_{\hat{\imath}^*} = \frac{-\mathcal{B}^* S_{\hat{\imath}}}{\mathcal{B}^n_{\hat{j}}} \quad .
\end{aligned}
\tag{5.16}
$$

Note that use of the similarity reduction as in equations (5.15-5.16) eliminates the need for applying a boundary condition at $\mathcal{B}_{min}, \mathcal{B}_{max}$. We can exclude the case $\mathcal{B}^n_{\hat{j}} = 0$ since (from equation (5.9))

$$
|\mathcal{B}^n_{\hat{j}}| = |\mathcal{B}^*|(1 + O(\Delta\tau)) \ . \tag{5.17}
$$

**Remark 5.1** (Reduction to a Two Dimensional Problem). *We can proceed more formally to eliminate the variable $\mathcal{B}$. If the similarity reduction (3.12) is valid, then we can define a function $\chi(z, \alpha, \tau)$ such that*

$$
\begin{aligned}
V(S, \mathcal{B}, \alpha, \tau) &= \mathcal{B}^2 \chi(S/\mathcal{B}, \alpha, \tau) \\
&= \mathcal{B}^2 \chi(z, \alpha, \tau) \\
& z_{\min} \le z \le z_{\max} \quad ; \quad z = \frac{S}{\mathcal{B}}
\end{aligned}
\tag{5.18}
$$

*Substituting equation (5.18) into equation (3.2) with payoff (3.3) gives an HJB equation for $\chi(z, \alpha, \tau)$. However, we will not follow this approach here. From an implementation point of view, application of the similarity reduction is simply a special (trivial) case of a full three dimensional implementation. There is no need for a separate implementation to handle the cases where the similarity reduction is valid/invalid. In addition, it is convenient to deal with the* physical *variables $(S, \mathcal{B}, \alpha)$, when dealing with boundary conditions, price impact functions and so on. Finally, our convergence proofs are given for the case of the similarity reduction. However, since we use the variables $(S, \mathcal{B}, \alpha, \tau)$, these proofs can be easily extended to the case where the similarity reduction is not valid.*

*The one complicating factor resulting from not carrying out the formal reduction to a two dimensional problem concerns the appropriate set of test functions to use in defining consistency in the viscosity solution sense. Since the problem is inherently two dimensional, this means that the test functions should be smooth, differentiable functions $\psi(z, \alpha, \tau)$. We cannot use arbitrary three dimensional test functions $\phi(S, \mathcal{B}, \alpha, \tau)$, but* *in view of equation (5.18), (which we use to define the interpolation operators (5.15-5.16)) we should use test functions of the form*

$$\phi(S, \mathcal{B}, \alpha, \tau) \quad = \quad \mathcal{B}^2 \psi(S/\mathcal{B}, \alpha, \tau) \quad . \tag{5.19}$$

*Let* $\mathbf{x} = (S, \mathcal{B}, \alpha, \tau)$*, then we can write equation (5.19) as*

$$\phi = \phi(\mathbf{x}) = \phi(\mathbf{x}, \psi(\mathbf{x})) = \phi(\mathbf{x}, \psi(S/\mathcal{B}, \alpha, \tau)) \quad . \tag{5.20}$$

## 5.2 Solution of the Local Optimization Problem

Recall equation (5.11)

$$V_{i,j,k}^{n+1} \quad = \quad \min_{v_{i,j,k}^{n+1} \in Z_{i,j,k}^{n+1}} V_{\hat{i},\hat{j},\hat{k}}^{n} + \Delta\tau (\mathcal{L}_h V)_{i,j,k}^{n+1} \quad . \tag{5.21}$$

An obvious way to solve the local optimization problem is to use a standard one-dimensional algorithm. However, we found this to be unreliable, since the local objective function has multiple local minima (this will be discussed in more detail later). Instead, we discretize the range of controls. For example, consider the set of controls $Z = [v_{\min}, v_{\max}]$ for a point in the interior of the computational domain. Let $\hat{Z} = \{v_0, v_1, ..., v_k\}$ with $v_0 = v_{\min}, v_k = v_{\max}$ and $\max_i v_{i+1} - v_i = O(h)$. Then, if $\phi$ is a smooth test function and $f(v), g(v)$ are continuous functions (which we assume to be the case) then

$$\begin{aligned} &\Bigg| \phi_\tau - \mathcal{L}\phi - r\mathcal{B}\phi_{\mathcal{B}} - \min_{v \in \hat{Z}} \Big[ -vSf(v)\phi_{\mathcal{B}} + v\phi_\alpha + g(v)S\phi_S \Big] \\ &- \Bigg( \phi_\tau - \mathcal{L}\phi - r\mathcal{B}\phi_{\mathcal{B}} - \min_{v \in Z} \Big[ -vSf(v)\phi_{\mathcal{B}} + v\phi_\alpha + g(v)S\phi_S \Big] \Bigg) \Bigg| \\ &\to 0 \quad ; \quad \text{as } h \to 0 \quad . \end{aligned} \tag{5.22}$$

Consequently, replacing $Z$ by $\hat{Z}$ is a consistent approximation [33]. Our actual numerical algorithm uses $Z_{i,j,k}^{n+1} \subseteq \hat{Z}$, and the minimum in equation (5.21) is found by linear search. Note that this approximation would be $O(h)$ if $f(v), g(v)$ are Lipshitz continuous.

# 6 Convergence to the Viscosity Solution

Provided a strong comparison result for the PDE applies, [7, 5] demonstrate that a numerical scheme will converge to the viscosity solution of the equation if it is $l_\infty$ stable, monotone, and pointwise consistent. In Appendix A, we prove the convergence of our numerical scheme (5.11) to the viscosity solution of problem (4.1) associated with boundary conditions (4.4-4.5), (4.10) by verifying these three properties.

The definition of consistency in the viscosity solution sense [5] appears to be somewhat complex. However, as can be seen in Appendix A, this definition is particularly useful in the context of a semi-Lagrangian discretization, since there are nodes in strips near the boundaries where the discretization is not consistent in the classical sense for arbitrary mesh/timestep sizes.

# 7 Optimal Liquidation Example: Short Trading Horizon

We use the parameters shown in Table 7.1, for an example where the entire stock position is to be liquidated in one day. Equations (3.2) and (3.4) are solved numerically using a semi-Lagrangian method described in Section 5. A similarity reduction is used to reduce the problem to a two dimensional $S \times \alpha$ grid, with two nodes (for all mesh/timestep sizes) in the $\mathcal{B}$ direction, as described in Section 5.1.

Table 7.2 shows the number of nodes and timesteps used in the convergence study. Table 7.3 shows the value of $E_{v^*}^{t=0}[\mathcal{B}_L^2]$ at $t = 0$, $S = 100$, $\alpha = 1$, $\mathcal{B} = -100$ for several levels of refinement. Convergence appears to be at a first order rate. Increasing the size of $S_{\max}$ resulted in no change to the solution to eight digits.

| Parameter | Value |
|---|---|
| $\sigma$ | 1.0 |
| $T$ | 1/250 years |
| $\eta$ | 0.0 |
| $r$ | 0.0 |
| $S_0$ | 100 |
| $\alpha_I$ | 1.0 |
| $\kappa_p$ | 0.0 |
| $\kappa_t$ | $2 \times 10^{-6}$ |
| $\kappa_s$ | 0.0 |
| $\beta$ | 1.0 |
| Action | Sell |
| $v_{min}$ | -1000/T |
| $v_{max}$ | 0.0 |
| $S_{max}$ | 20000 |
| $(\Delta t)_T$ (2.12) | $10^{-6}$ years |

TABLE 7.1: *Parameters for optimal execution example, short trading horizon.*

| Timesteps | $S$ nodes | $\alpha$ nodes | $\mathcal{B}$ nodes | v nodes | Refinement Level |
|---|---|---|---|---|---|
| 25 | 98 | 41 | 77 | 30 | 0 |
| 50 | 195 | 81 | 153 | 59 | 1 |
| 100 | 389 | 161 | 305 | 117 | 2 |
| 200 | 777 | 321 | 609 | 233 | 3 |
| 400 | 1553 | 641 | 1217 | 465 | 4 |

TABLE 7.2: *Grid and timestep data for convergence studies. If a similarity reduction is used, then the $\mathcal{B}$ grid has only two nodes for any refinement level.*

| Refinement Level | Value |
|---|---|
| 0 | 1.668460 |
| 1 | 1.319408 |
| 2 | 1.176402 |
| 3 | 1.094543 |
| 4 | 1.054693 |

TABLE 7.3: *Value of $E^{t=0}_{v^*}[\mathcal{B}_L^2]$ at $t = 0$, $S = 100$, $\alpha = 1$, $\mathcal{B} = -100$. Data in Table 7.1. Discretization data is given in Table 7.2.*

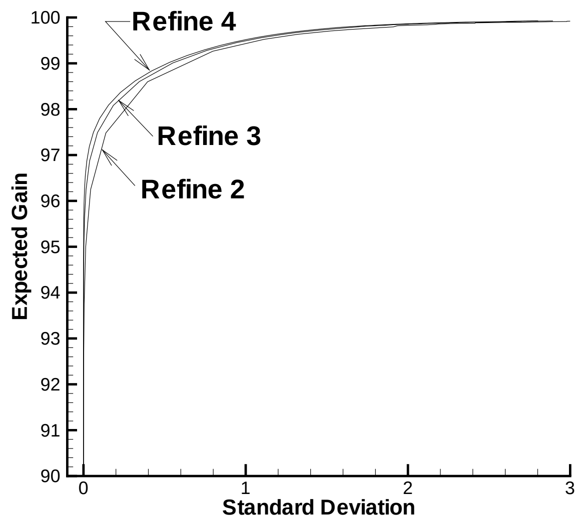


FIGURE 7.1: *The efficient frontier for optimal execution (sell case), using the data in Table 7.1. The vertical axis represents the expected average share price obtained. Initial stock price $S_0 = 100$. Discretization details given in Table 7.2. Similarity reduction used.*

The efficient frontier is shown in Figure 7.1. This Figure shows the expected average amount obtained per share versus the standard deviation. The pre-trade share price is \$100. The results in Figure 7.1 were obtained using the similarity reduction.

For comparative purposes, we also show the efficient frontier in Figure 7.2, obtained using the full three dimensional PDE (no similarity reduction). Due to memory requirements, we can only show three levels of refinement. Note that the full three dimensional PDE uses a discretization in the $\mathcal{B}$ direction. Recall that the use of a similarity reduction (as described in Section 3.4) effectively means that there is no discretization error in the $\mathcal{B}$ direction. Hence we can expect that the full three dimensional PDE solve will show larger discretization errors, compared to the solution obtained using the similarity reduction, for the same refinement level. As shown in Figure 7.2, the full three dimensional solution is converging to the same efficient frontier as the similarity reduction solution, but more slowly and at much greater computational cost.

Figure 7.3 shows $E_{v^*}^{t=0}[\mathcal{B}_L^2]$, $\mathcal{B} = -100$. This value of $\mathcal{B} = -100$ corresponds to $\gamma = 200$. Assuming we are at the initial point $(S = 100, B = 0, \alpha = 1)$, this value of $\gamma$ corresponds to the point

$$\begin{aligned} \text{Expected Gain} &= 99.295 \\ \text{Standard Deviation} &= 0.7469 \end{aligned} \tag{7.1}$$

on the curve shown in Figure 7.1.

## 7.1 Optimal Strategy: Uniqueness

From Figure 7.3 we can see that there is a large region for $S > 100$ where

$$V_\alpha \simeq 0 \;\; ; \;\; V_S \simeq 0 \;\; ; \;\; V \simeq 0 \tag{7.2}$$

which then implies, using equation (4.6), that $V_{\mathcal{B}} \simeq 0$. Hence, in the flat region in Figure 7.3, $V_\alpha \simeq 0$, $V_S \simeq 0$, and $V_{\mathcal{B}} \simeq 0$.

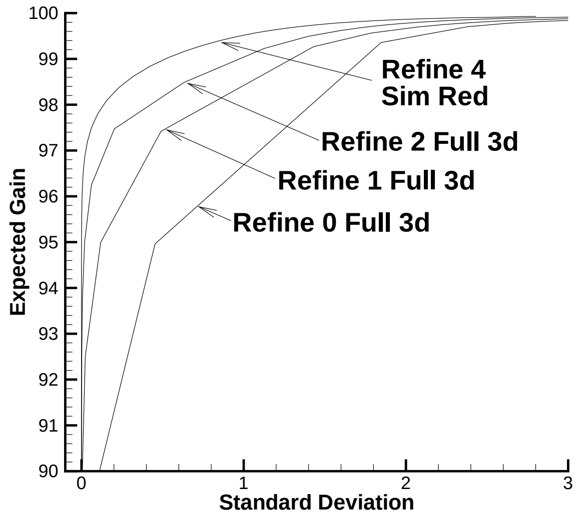


Figure 7.2: *The efficient frontier for optimal execution (sell case), using the data in Table 7.1. The vertical axis represents the expected average share price obtained. Initial stock price* $S_0 = 100$*. Discretization details given in Table 7.2. Results are obtained by solving the full three dimensional PDE. The curve labelled "Sim Red" was computed using the similarity reduction method (as in Figure 7.1).*

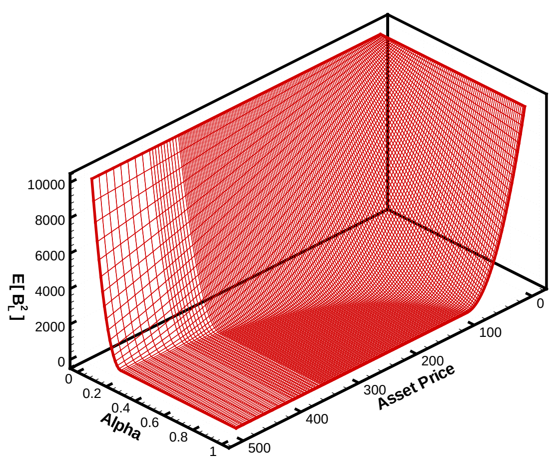


Figure 7.3: *The value surface* $E_{v^*}^{t=0}[\mathcal{B}_L^2]$*,* $\mathcal{B} = -100$*,* $t = 0$*. Data in Table 7.1.*

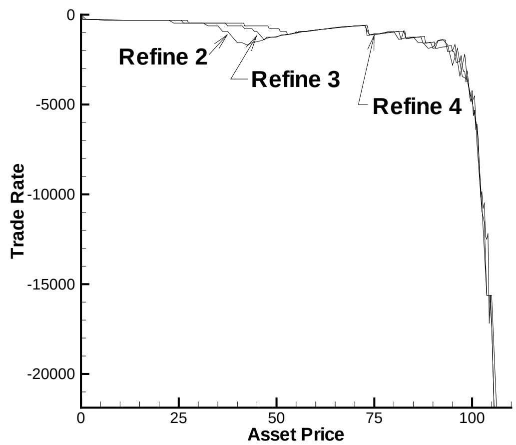


FIGURE 7.4: *Optimal trading rate at $t = 0.0$, $B = 0$, $\alpha = 1$, as a function of $S$. This is the optimal strategy for the point on the efficient frontier given by equation (7.1). Note that the constant trading rate which meets the liquidation objective is $v = -250$. Data in Table 7.1. Discretization details given in Table 7.2.*

Recall equation (3.2)

$$V_\tau \quad = \quad \mathcal{L}V + r\mathcal{B}V_{\mathcal{B}} + \min_{v \in [v_{min}, v_{max}]} \left[ -vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S \right] \ . \tag{7.3}$$

If $V_S = V_{\mathcal{B}} = V_\alpha = 0$, then the optimal control can be any value $v \in [v_{min}, v_{max}]$. Clearly there are large regions where the optimal strategy is not unique.

As an extreme example, one way to achieve minimal risk is to immediately sell all stock at an infinite rate, which results in zero expected gain, and zero standard deviation. However, this strategy is not unique. Another possibility is to do nothing until $t = T^-$, and then to sell at an infinite rate. This will also result in zero gain and zero standard deviation. There are infinitely many strategies which produce the identical result. Hence, in general, the optimal strategy is not unique, but the value function is unique.

## 7.2 Optimal Trading Strategy

Figure 7.4 shows the optimal trading rate at $t = 0.0$, $\mathcal{B} = -100$, $\alpha = 1$, as a function of $S$. This is the optimal strategy for the point on the efficient frontier given by equation (7.1). We can interpret this curve as follows. Given the initial data ($S = 100, \alpha = 1, B = 0, t = 0$), this curve shows the optimal trading rate if the asset price suddenly changes to the value of $S$ shown. Note that this particular strategy is the rate which minimizes (2.16) for the value of $\gamma$ which results in (7.1). To put Figure 7.4 in perspective, the constant trading rate which meets the liquidation objective is $v = -1/T = -250$.

The optimal trading rate behaves roughly as expected [28]. As the asset price increases, the trading rate should also increase. In other words, some of the unexpected gain in stock price can be spent to reduce the standard deviation. Recall that the strategy maximizes (2.16) as seen at the initial time.

However, note the sawtooth pattern in the optimal trading rate for $S > 75$. This does not appear to be an artifact of the discretization, since this pattern seems to persist for small mesh sizes.

It is perhaps not immediately obvious how a smooth value function as given in Figure 7.3 can produce the non-smooth trading strategy shown in Figure 7.4. Recall that a local optimization problem (5.21) is solved at each node to determine the optimal trade rate. A careful analysis of the objective function at the points corresponding to the sawtooth pattern in Figure 7.4 revealed that the value function was very

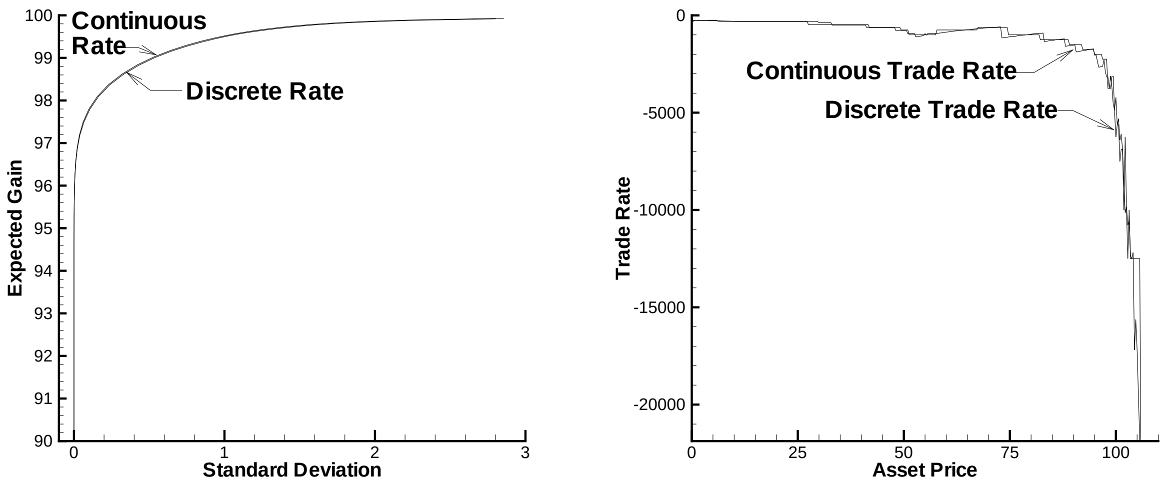


FIGURE 7.5: *Left plot: the efficient frontier for optimal execution (sell case), using the data in Table 7.1. The vertical axis represents the expected average share price obtained. Initial stock price* $S_0 = 100$. *The curves are computed with refinement level 4 (see Table 7.2). The two curves are computed using the set of trade rates in equation (7.4) (Discrete Trade Rate), and the approximation to continuous trading rates obtained by discretizing* $[v_{min}, v_{max}]$ *with* 465 *nodes (Continuous Trade Rate). Right plot: the optimal trading rates corresponding to the efficient frontiers in the left plot.*

flat, with multiple local minima. Although the value function is a smooth function of $S$, the optimal trade amount $(v\Delta t)$ is not a smooth function of $S$.

This suggests that the optimal value is not very sensitive to the control at these points.

## 7.3 Discrete Trade Rates

In order to explore the effect of the sawtooth pattern on the optimal trade rates, the optimal strategy was recomputed using a fixed number of discrete trading rates. The rates were (in units of $1/T$)

$$\begin{aligned}\text{Trade rates } = \{&-1000, -500., -100., -50., -40., -30.,\\ &-25, -20., -15., -10., -9., -8., -7., -6.,\\ &-5., -4.5, -4., -3.5, -3., -2.5, -2., -1.5,\\ &-1.25, -1.0, -.75, -.5, -.25, 0.\}\end{aligned} \tag{7.4}$$

These discrete trade rates were fixed, and not changed for finer grids. Recall that for the *continuous* case, the spacing of the discrete trading rates was divided by two on each grid refinement. On the finest grid $(1553 \times 641)$ the interval $[-v_{min}, v_{max}]$ was discretized using 465 nodes. Note that there are only 27 discrete trading rates in the set of nodes in equation (7.4). The efficient frontier using both these possible sets of trading rates is shown in Figure 7.5 (left plot). The two curves are almost indistinguishable.

This has an interesting practical benefit. If $h$ is the mesh/timestep size parameter (see equation (A.1)), then the method developed here has complexity $O(1/h^4)$. One might expect a complexity of $O(1/h^3)$ but the need to solve the local optimization problem using a linear search generates the extra power of $1/h$. However, from Figure 7.5, it would appear that we can determine the efficient frontier to a practical level of accuracy using a mesh independent set of trading rates, which would lower the complexity to $O(1/h^3)$.

Figure 7.5 (right plot) also shows the optimal trading rates corresponding to the efficient frontiers shown in Figure 7.5 (left plot). It would appear that there are many strategies which generate very similar efficient

| Parameter | Value |
|---|---|
| $\sigma$ | .40 |
| $T$ | 1/12 years |
| $\eta$ | .10 |
| $r$ | 0.05 |
| $S_0$ | 100 |
| $\alpha_{sell}$ | 1.0 |
| $\kappa_p$ | 0.01 |
| $\kappa_t$ | .069 |
| $\kappa_s$ | 0.01 |
| $\beta$ | .5 |
| Action | Sell |
| $v_{min}$ | -25/T |
| $v_{max}$ | 0.0 |
| $S_{max}$ | 20000 |
| $(\Delta t)_T$ (2.12) | $10^{-9}$ years |

TABLE 8.1: *Parameters for optimal execution example, long trading horizon.*

frontiers. It is likely that the sawtooth pattern in Figure 7.4 is due to the ill-posed nature of the optimal strategy.

# 8 Liquidation Example: Long Trading Horizon

Table 8.1 shows the data used for a second example. Note that $\beta$ in equation (2.10) is set to $\beta = .5$. Similar values of $\beta$ have been reported in [25].

Figure 8.1 shows the efficient frontier. Figure 8.2 shows the the optimal trading rate at $t = 0.0$, $\mathcal{B} = -100$, $\alpha = 1$, as a function of $S$. The trade rates are given for a point on the efficient frontier corresponding to ($\gamma = 200.83$)

$$\begin{aligned} \text{Expected Gain} &= 95.6 \\ \text{Standard Deviation} &= 3.47 \ . \end{aligned} \tag{8.1}$$

Once again, we see that the efficient frontier is smooth, but that the optimal trading rates show the same sawtooth pattern as observed in Figure 7.4. This indicates that the optimal trading rates are somewhat ill posed.

# 9 Conclusion

We have formulated the problem of determining the efficient frontier (and corresponding optimal strategy) in terms of an equivalent LQ problem. We need only solve a single nonlinear HJB equation (and an associated linear PDE) to construct the entire efficient frontier.

The HJB equation is discretized using a semi-Lagrangian approach. Assuming that the HJB equation satisfies a strong comparison property, then we have proven convergence to the viscosity solution by showing that the scheme is monotone, consistent and stable. Note that in this case, it is useful to use consistency in the viscosity solution sense [7, 5] since the semi-Lagrangian method is not classically consistent (for arbitrary grid sizes) at points near the boundaries of the computational domain.

The semi-Lagrangian discretization separates the model of the underlying stochastic process from the model of price impact. Changing the particular model of price impact amounts to changing a single function in the implementation. The semi-Lagrangian method is also highly amenable to parallel implementation.

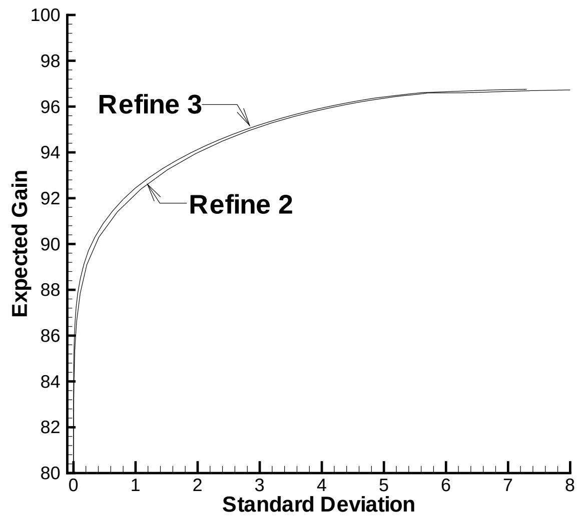


FIGURE 8.1: *The efficient frontier for optimal execution (sell case), using the data in Table 8.1. The vertical axis represents the expected average share price obtained. Initial stock price $S_0 = 100$. Discretization details given in Table 7.2.*

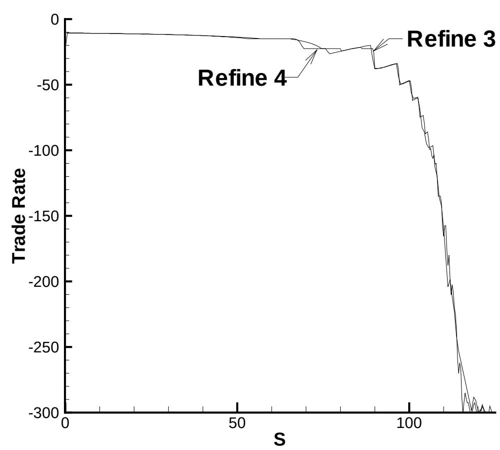


FIGURE 8.2: *Optimal trading rate at $t = 0.0$, $B = 0$, $\alpha = 1$, as a function of $S$. This is the optimal strategy for the point on the efficient frontier given by equation (8.1). Note that the constant trading rate which meets the liquidation objective is $v = -12$. Data in Table 8.1. Discretization details given in Table 7.2.*

The efficient frontiers computed using the method developed in this work are consistent with intuition. However, the optimal trading rates, as a function of the asset price at the initial time, show an unexpected sawtooth pattern for large asset prices. A detailed analysis of the numerical results shows that that there are many strategies which give virtually the same value function. Hence, the numerical problem for the optimal strategy (as opposed to the efficient frontier) appears to be ill-posed. Note that this ill-posedness seems to be a particular property of the pre-commitment mean-variance objective function, and is not seen if alternative objective functions are used, such as a utility function [31] or mean-quadratic variation [19].

However, this ill-posedness in terms of the strategy is not particularly disturbing in practice. The end result is that there are many strategies which give essentially the same efficient frontier, which is the measure of practical importance. This also indicates that it is possible to vary the trading rates in an unpredictable pattern, which may be useful to avoid signalling trading strategies, yet still achieve a mean variance efficient result.

# A Convergence to the Viscosity Solution of (4.1)

In this Appendix, we will verify that the discrete scheme (5.11) is consistent, stable and monotone, which ensures convergence to the viscosity solution of (4.1) associated with boundary conditions (4.4-4.5), (4.10). We will assume that the similarity reduction equations (5.15) and (5.16) are used in the following analysis.

## A.1 Some Preliminary Results

It will be convenient to define $\Delta S_{\max} = \max_i \big(S_{i+1} - S_i\big)$, $\Delta S_{\min} = \min_i \big(S_{i+1} - S_i\big)$, $\Delta\alpha_{\max} = \max_j \big(\alpha_{k+1} - \alpha_k\big)$, $\Delta\alpha_{\min} = \min_k \big(\alpha_{k+1} - \alpha_k\big)$. We assume that there is a mesh size/timestep parameter $h$ such that

$$\Delta S_{\max} = C_1 h \;\; ; \;\; \Delta\alpha_{\max} = C_2 h \;\; ; \;\; \Delta\tau = C_3 h \;\; ; \;\; \Delta S_{\min} = C_1' h \;\; ; \;\; \Delta\alpha_{\min} = C_2' h. \tag{A.1}$$

where $C_1, C_1', C_2, C_2', C_3$ are constants independent of $h$.

If test function $\phi$ is of the form (5.19-5.20), then we can write

$$\phi(S, \mathcal{B}, \alpha, \tau, \psi(S, \mathcal{B}, \alpha, \tau)) \;\; = \;\; \mathcal{B}^2 \psi(S/\mathcal{B}, \alpha, \tau) \;\; . \tag{A.2}$$

where we assume that $\psi(S/\mathcal{B}, \alpha, \tau) = \psi(z, \alpha, \tau)$ is a smooth function of $(z, \alpha, \tau)$, which has bounded derivatives with respect to $(z, \alpha, \tau)$ on $[z_{\min}, z_{\max}] \times [\alpha_{\min}, \alpha_{\max}] \times [0, T]$. Note that since $|B_j| > 0$, and $B_{\hat{j}} = B_j(1 + O(h))$, then $\phi$ has bounded derivatives with respect to $(S, \mathcal{B}, \alpha, \tau)$ for $\mathcal{B}$ near $\mathcal{B}_0, \mathcal{B}_1$, for $h$ sufficiently small, since $\psi$ has bounded derivatives with respect to $(z, \alpha, \tau)$.

For more compact notation, we will also define

$$\begin{aligned} \mathbf{x}_{i,j,k}^n &= (S_i, \mathcal{B}_j, \alpha_k, \tau^n) \\ \phi(S, \mathcal{B}, \alpha, \tau, \psi(S, \mathcal{B}, \alpha, \tau)) &= \phi(\mathbf{x}, \psi(\mathbf{x})) \\ \phi_{i,j,k}^n &= \phi(\mathbf{x}_{i,j,k}^n) = \phi(\mathbf{x}_{i,j,k}^n, \psi(\mathbf{x}_{i,j,k}^n)) \;\; . \end{aligned} \tag{A.3}$$

Taylor series (see [13]) gives

$$(\mathcal{L}_h \phi)_{i,j,k}^n = (\mathcal{L}\phi)_{i,j,k}^n + O(h) \;\; . \tag{A.4}$$

and if $\xi$ is a constant, we also have (noting equation (A.2))

$$\phi(\mathbf{x}, \psi(\mathbf{x}) + \xi)_{i,j,k}^n \;\; = \;\; \phi_{i,j,k}^n + \mathcal{B}_j^2 \xi \;\; , \tag{A.5}$$

and

$$(\mathcal{L}_h(\phi(\mathbf{x}, \psi + \xi))_{i,j,k}^n = (\mathcal{L}\phi)_{i,j,k}^n + O(h) \;\; . \tag{A.6}$$

Assuming $\phi$ is of the form (A.2) and noting interpolation scheme (5.15-5.16) we obtain, using equations (5.8-5.9)

$$
\begin{aligned}
&\phi^n_{\hat{i},\hat{j},\hat{k}} \\
&\quad = \phi\Bigg( S_i \exp[g(v^{n+1}_{i,j,k}\Delta\tau], \mathcal{B}_j \exp[r\Delta\tau] - v^{n+1}_{i,j,k} S_i f(v^{n+1}_{i,j,k}) \left( \frac{e^{r\Delta\tau} - e^{g(v^{n+1}_{i,j,k})\Delta\tau}}{r - g(v^{n+1}_{i,j,k})} \right) \alpha_k + v^{n+1}_{i,j,k}\Delta\tau, \tau^n \Bigg) \\
&\qquad + O(h^2) \\
&\quad = \phi\Bigg( S_i + S_i g(v^{n+1}_{i,j,k})\Delta\tau, \mathcal{B}_j + (r\mathcal{B}_j - v^{n+1}_{i,j,k} S_i f(v^{n+1}_{i,j,k}))\Delta\tau, \alpha_k + v^{n+1}_{i,j,k}\Delta\tau, \tau^n \Bigg) + O(h^2) \quad .
\end{aligned}
\tag{A.7}
$$

Noting that

$$
\left( \frac{\mathcal{B}^n_{\hat{j}}}{\mathcal{B}_j} \right)^2 = 1 + O(h) \tag{A.8}
$$

and that if $\xi$ is a constant, then the linear interpolation in equation (5.15-5.16) is exact for constants, then we obtain

$$
\begin{aligned}
&\phi(\mathbf{x}, \psi(\mathbf{x}) + \xi)^n_{\hat{i},\hat{j},\hat{k}} = \\
&\quad \phi\Bigg( S_i + S_i g(v^{n+1}_{i,j,k})\Delta\tau, \mathcal{B}_j + (r\mathcal{B}_j - v^{n+1}_{i,j,k} S_i f(v^{n+1}_{i,j,k}))\Delta\tau, \alpha_k + v^{n+1}_{i,j,k}\Delta\tau, \tau^n \Bigg) \\
&\quad + O(h^2) + \mathcal{B}_j^2 \xi (1 + O(h))
\end{aligned}
\tag{A.9}
$$

## A.2 Stability

**Definition A.1** ($l_\infty$ stability). *Discretization (5.11) is $l_\infty$ stable if*

$$
\|V^{n+1}\|_\infty \quad \leq \quad C_4 \quad , \tag{A.10}
$$

*for $0 \leq n \leq N-1$ as $h \to 0$, where $C_4$ is a constant independent of $h$. Here* $\|V^{n+1}\|_\infty = \max_{i,j,k} |V^{n+1}_{i,j,k}|$.

**Lemma A.1** ($l_\infty$ stability). *If the discretization (5.4) satisfies the positive coefficient condition (5.5) and linear interpolation is used to compute $V^n_{\hat{i},\hat{j},\hat{k}}$, then the scheme (5.11) with payoff (5.12), using the similarity reduction (5.15-5.16), satisfies*

$$
\|V^n\|_\infty \leq e^{2rT} \|V^0\|_\infty \tag{A.11}
$$

*for $0 \leq n \leq N = T/\Delta\tau$ as $h \to 0$.*

*Proof.* First, note that from payoff condition (5.12) we have $0 \leq V^0_{i,j,k} \leq \|\mathcal{B}_L^2\|_\infty$, which is bounded since the computational domain is bounded.

Now, suppose that

$$
0 \leq V^n_{i,j,k} \leq \|V^n\|_\infty \quad . \tag{A.12}
$$

Define

$$
V^{n+}_{i,j,k} \quad = \quad \min_{v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}} V^n_{\hat{i},\hat{j},\hat{k}} \quad . \tag{A.13}
$$

Since linear interpolation is used, then from equation (A.12), $V^{n+}_{i,j,k} \geq 0$. Since $v^{n+1}_{i,j,k} = 0 \in Z^{n+1}_{i,j,k}$, then from equations (5.8), (5.15-5.16) and the fact that linear interpolation is used to compute $V^n_{\hat{i}^*,j,\hat{k}}$, we have that $0 \leq V^{n+}_{i,j,k} \leq e^{2r\Delta\tau}\|V^n\|_\infty$.

Since discretization (5.4) is a positive coefficient method, a straightforward maximum analysis shows that

$$
\begin{aligned}
0 \le V_{i,j,k}^{n+1} &\le \|V^{n+}\|_\infty \\
&\le e^{2r\Delta\tau}\|V^n\|_\infty \le e^{2rT}\|V^0\|_\infty \ .
\end{aligned}
\tag{A.14}
$$

□

## A.3 Consistency

Let

$$
\begin{aligned}
&\mathcal{H}_{i,j,k}^{n+1}\left(h, V_{i,j,k}^{n+1}, \{V_{l,m,p}^{n+1}\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \{V_{i,j,k}^n\}\right) \\
&= \frac{1}{\Delta\tau}\left[V_{i,j,k}^{n+1} - \min_{v_{i,j,k}^{n+1}\in Z_{i,j,k}^{n+1}} V_{\hat{i},\hat{j},\hat{k}}^n - \Delta\tau(\mathcal{L}_h V)_{i,j,k}^{n+1}\right]
\end{aligned}
\tag{A.15}
$$

where

$$
\{V_{l,m,p}^{n+1}\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}
\tag{A.16}
$$

is the set of values $V_{l,m,p}^{n+1}$, $l \neq i$, $l = 0,\ldots,i_{\max}$ and $m \neq j$, $m = 0,\ldots,j_{\max}$, $p \neq k$, $p = 0,\ldots,k_{\max}$, and $\{V_{i,j,k}^n\}$ is the set of values $V_{i,j,k}^n$, $i = 0,\ldots,i_{\max}$, $j = 0,\ldots,j_{\max}$, $k = 0,\ldots,k_{\max}$.

We can then define the complete discrete scheme as

$$
\begin{aligned}
&\mathcal{G}_{i,j,k}^{n+1}\big(h, V_{i,j,k}^{n+1}, \{V_{l,m,p}^{n+1}\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \{V_{i,j,k}^n\}\big) \\
&\equiv \begin{cases} \mathcal{H}_{i,j,k}^{n+1} & \text{if } \ 0 \le S_i \le S_{i_{\max}}, \quad \mathcal{B}_j \in \mathcal{B}_{set}, \quad \alpha_{min} \le \alpha_k \le \alpha_{max}, \quad 0 < \tau^{n+1} \le T \\ V_{i,j,k}^{n+1} - \big((\mathcal{B}_L)_{i,j,k}\big)^2 & \text{if } \ 0 \le S_i \le S_{i_{\max}}, \quad \mathcal{B}_j \in \mathcal{B}_{set}, \quad \alpha_{min} \le \alpha_k \le \alpha_{max}, \quad \tau^{n+1} = 0 \end{cases} \\
&= 0 \ .
\end{aligned}
\tag{A.17}
$$

**Remark A.1.** *We have written equation (A.15) as if we find the exact minimum at each node. In practice, we find the approximate minimum as described in Section 5.2. To avoid notational complexity, we will carry out our analysis assuming the algorithm determines the exact minimum. However, in view of equation (5.22), the use of the approximate minimum is a consistent approximation to the original problem, as long as the node spacing in* $[v_{\min}, v_{\max}]$ *tends to zero as* $h \to 0$ *[33].*

Let $\Omega$ be the set of points $(S, \mathcal{B}, \alpha, \tau)$ such that $\Omega = [0, S_{\max}] \times \mathcal{B}_{set} \times [\alpha_{min}, \alpha_{max}] \times [0, T]$. The domain $\Omega$ can divided into the subregions

$$
\begin{aligned}
\Omega_{in} &= [0, S_{\max}) \times \mathcal{B}_{set} \times (\alpha_{\min}, \alpha_{\max}) \times (0, T] \\
\Omega_{\alpha_{\min}} &= [0, S_{\max}) \times \mathcal{B}_{set} \times \{\alpha_{\min}\} \times (0, T] \\
\Omega_{\alpha_{\max}} &= [0, S_{\max}) \times \mathcal{B}_{set} \times \{\alpha_{\max}\} \times (0, T] \\
\Omega_{S_{\max}} &= \{S_{\max}\} \times \mathcal{B}_{set} \times (\alpha_{\min}, \alpha_{\max}) \times (0, T] \\
\Omega_{S_{\max}\alpha_{\min}} &= \{S_{\max}\} \times \mathcal{B}_{set} \times \{\alpha_{\min}\} \times (0, T] \\
\Omega_{S_{\max}\alpha_{\max}} &= \{S_{\max}\} \times \mathcal{B}_{set} \times \{\alpha_{\max}\} \times (0, T] \\
\Omega_{\tau^0} &= [0, S_{\max}] \times \mathcal{B}_{set} \times [\alpha_{\min}, \alpha_{\max}) \times \{0\},
\end{aligned}
\tag{A.18}
$$

where $\Omega_{in}$ represents the interior region, and $\Omega_{\alpha_{\min}}, \Omega_{\alpha_{\max}}, \Omega_{S_{\max}}, \Omega_{\tau^0}, \Omega_{S_{\max}\alpha_{\max}}, \Omega_{S_{\max}\alpha_{\min}}$ denote the boundary regions. If $\mathbf{x} = (S, \mathcal{B}, \alpha, \tau)$, let $DV(\mathbf{x}) = (V_S, V_{\mathcal{B}}, V_\alpha, V_\tau)$ and $D^2V(\mathbf{x}) = V_{SS}$. Let us define the following operators:

$$
\begin{aligned}
F_{in}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - \mathcal{L}V - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\Big] \\
F_{\alpha_{\min}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - \mathcal{L}V - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z^+}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\Big] \\
F_{\alpha_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - \mathcal{L}V - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z^-}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\Big] \\
F_{S_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha\Big] \\
F_{S_{\max}\alpha_{\min}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z^+}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha\Big] \\
F_{S_{\max}\alpha_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z^-}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha\Big] \\
F_{\tau^0}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V - \mathcal{B}_L^2
\end{aligned}
\tag{A.19}
$$

Then the problem (4.1-4.10) can be combined into one equation as follows:

$$
F\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) = 0 \quad \text{for all } \mathbf{x} = (S, \mathcal{B}, \alpha, \tau) \in \Omega\ , \tag{A.20}
$$

where $F$ is defined by

$$
F = \begin{cases}
F_{in}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{in}, \\
F_{\alpha_{\min}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{\alpha_{\min}}, \\
F_{\alpha_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{\alpha_{\max}}, \\
F_{S_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{S_{\max}}, \\
F_{S_{\max}\alpha_{\max}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{S_{\max}\alpha_{\max}}, \\
F_{S_{\max}\alpha_{\min}}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{S_{\max}\alpha_{\min}}, \\
F_{\tau^0}\big(V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{\tau_0}.
\end{cases}
\tag{A.21}
$$

In order to demonstrate consistency, we first need some intermediate results. For given $\Delta\tau$, consider the continuous form of equations (5.8)

$$
\begin{aligned}
\hat{S} &= S\exp[g(v)\Delta\tau] \\
\hat{\mathcal{B}} &= \mathcal{B}\exp[r\Delta\tau] - vSf(v)\left(\frac{e^{r\Delta\tau} - e^{g(v)\Delta\tau}}{r - g(v)}\right) \\
\hat{\alpha} &= \alpha + v\Delta\tau \\
& \quad\ v \in [v_{\min}, v_{\max}]\quad .
\end{aligned}
\tag{A.22}
$$

Consider the domain

$$
\Omega_{Z'}(\Delta\tau) \subseteq [0, S_{\max}] \times \mathcal{B}_{set} \times (\alpha_{\min}, \alpha_{\max}) \times (0, T] \tag{A.23}
$$

where $(\hat{S}, \hat{\alpha}) \notin [0, S_{\max}] \times [\alpha_{\min}, \alpha_{\max}]$. In other words, for points in $\Omega_{Z'}$, the range of possible values of $v$ in equation (A.22) would have to be restricted to less than the full range $[v_{\min}, v_{\max}]$ in order to ensure that

$$
0 \le \hat{S} \le S_{\max} \qquad \alpha_{\min} \le \hat{\alpha} \le \alpha_{\max}\quad . \tag{A.24}
$$

For example, the region

$$
\begin{aligned}
&\alpha_{\max} - v_{\max}\Delta\tau < \alpha < \alpha_{\max} \\
&\alpha_{\min} < \alpha < \alpha_{\min} - v_{\min}\Delta\tau\quad ,
\end{aligned}
\tag{A.25}
$$

will be in $\Omega_{Z'}$. In general, $\Omega_{Z'}$ will consist of small strips near the boundaries of $\Omega$.

We define the set $Z'(\mathbf{x}, h) \subseteq Z$ such that if $\mathbf{x} \in \Omega_{Z'}$, then $v \in Z'(\mathbf{x}, h)$ ensures that equation (A.24) is satisfied. We define the operator

$$\begin{aligned} & F_{Z'}\big(D^2 V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) \\ & \qquad = V_\tau - \mathcal{L}V - r\mathcal{B}V_{\mathcal{B}} - \min_{v \in Z'}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha + g(v)SV_S\Big] \quad ; \quad \mathbf{x} \in \Omega_{Z'}, S < S_{\max} \\ & \qquad = V_\tau - r\mathcal{B}V_{\mathcal{B}} \min_{v \in Z'}\Big[-vSf(v)V_{\mathcal{B}} + vV_\alpha\Big] \quad ; \quad \mathbf{x} \in \Omega_{Z'}, S = S_{\max} \quad . \end{aligned} \tag{A.26}$$

**Lemma A.2.** *For any smooth test function of the form*

$$\begin{aligned} \phi(\mathbf{x}, \psi(\mathbf{x}) &= \mathcal{B}^2 \psi(z, \alpha, \tau) \\ & z = \frac{S}{\mathcal{B}} \end{aligned} \tag{A.27}$$

*where $\psi$ has bounded derivatives with respect to $(z, \alpha, \tau)$ for $(S, \mathcal{B}, \alpha, \tau) \in \Omega$, and*

$$S_{imax-1} \quad < \quad S_{imax} e^{-g(v_{\max})\Delta\tau} \tag{A.28}$$

*then*

$$\begin{aligned} & \mathcal{G}_{i,j,k}^{n+1}\Big(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n+1}, \big\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{l,m,p}^{n+1}\big\}_{\substack{l \neq i \\ m \neq j \\ p \neq k}}, \big\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n}\big\}\Big) \\ & = \begin{cases} F_{in} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{in} \backslash \Omega_{Z'} \\ F_{\alpha_{\min}} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{\alpha_{\min}} \\ F_{\alpha_{\max}} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{\alpha_{\max}} \\ F_{S_{\max}} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{S_{\max}} \backslash \Omega_{Z'} \\ F_{S_{\max}\alpha_{\max}} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{S_{\max}\alpha_{\max}} \\ F_{S_{\max}\alpha_{\min}} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{S_{\max}\alpha_{\min}} \\ F_{Z'} + O(h) + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{Z'} \\ F_{\tau^0} + O(\xi) & \text{if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{\tau^0} \end{cases} \end{aligned} \tag{A.29}$$

*where $\xi$ is a constant, and $F_{in}, F_{\alpha_{\min}}, F_{\alpha_{\max}}, F_{S_{\max}}, F_{Z'}, F_{\tau^0}, F_{S_{\max}\alpha_{\max}}, F_{S_{\max}\alpha_{\min}}$ are functions of $(D^2\phi(\mathbf{x}), D\phi(\mathbf{x}), \phi(\mathbf{x}), \mathbf{x})$.*

**Remark A.2.** *Condition A.28 is a very mild restriction on the placement of node $S_{imax-1}$ and is not practically restrictive. This condition ensures that, for example, if $\mathbf{x}_{i,j,k}^{n+1} \in \Omega_{\alpha_{\min}}$ or $\mathbf{x}_{i,j,k}^{n+1} \in \Omega_{\alpha_{\max}}$, then $\mathbf{x}_{i,j,k}^{n+1} \notin \Omega_{Z'}$.*

*Proof.* Consider the case $\mathbf{x} \in \Omega_{in} \backslash \Omega_{Z'}$. From equations (A.4), (A.5), (A.6), (A.9), we obtain

$$
\begin{aligned}
&\frac{1}{\Delta\tau}\left[\phi(\mathbf{x}, \psi(\mathbf{x}) + \xi)_{i,j,k}^{n+1} - \min_{v_{i,j,k}^{n+1} \in Z_{i,j,k}^{n+1}} \phi(\mathbf{x}, \psi(\mathbf{x}) + \xi)_{\hat{i},\hat{j},\hat{k}}^{n} - \Delta\tau(\mathcal{L}_h(\phi(\mathbf{x}, \psi + \xi))_{i,j,k}^{n+1}\right] \\
&= \frac{1}{\Delta\tau}\left[\phi_{i,j,k}^{n+1} - \phi_{i,j,k}^{n} - \min_{v_{i,j,k}^{n+1} \in Z_{i,j,k}^{n+1}} \left\{ (\phi_S)_{i,j,k}^{n} S_i g(v_{i,j,k}^{n+1}) \Delta\tau \right.\right. \\
&\quad \left.\left. + (\phi_{\mathcal{B}})_{i,j,k}^{n} (r\mathcal{B}_j - v_{i,j,k}^{n+1} S_i f(v_{i,j,k}^{n+1}))\Delta\tau + (\phi_\alpha)_{i,j,k}^{n} v_{i,j,k}^{n+1} \Delta\tau + O(h^2) + O(h\xi) \right\}\right] \\
&\quad - (\mathcal{L}\phi)_{i,j,k}^{n+1} + O(h) \\
&= (\phi_\tau)_{i,j,k}^{n+1} - (\mathcal{L}\phi)_{i,j,k}^{n+1} - \min_{v_{i,j,k}^{n+1} \in Z_{i,j,k}^{n+1}} \left\{ (\phi_S)_{i,j,k}^{n+1} S_i g(v_{i,j,k}^{n+1}) + (\phi_{\mathcal{B}})_{i,j,k}^{n+1} (r\mathcal{B}_j - v_{i,j,k}^{n+1} S_i f(v_{i,j,k}^{n+1})) \right. \\
&\quad \left. + (\phi_\alpha)_{i,j,k}^{n} v_{i,j,k}^{n+1} + O(\xi) + O(h) \right\} + O(h) \\
&= \left[ \phi_\tau - \mathcal{L}\phi - \min_{v \in Z} \left\{ \phi_S S g(v) + \phi_{\mathcal{B}}(r\mathcal{B} - vSf(v)) + \phi_\alpha v \right\} \right]_{i,j,k}^{n+1} + O(\xi) + O(h) \quad .
\end{aligned}
\tag{A.30}
$$

where we have taken the $O(h), O(\xi)$ terms out of the min since they are bounded functions of $v_{i,j,k}^{n+1}$ (see [12]). As a result, we have

$$
\begin{aligned}
&\mathcal{G}_{i,j,k}^{n+1}\left(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n+1}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{l,m,p}^{n+1}\right\}_{\substack{l \neq i \\ m \neq j \\ p \neq k}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n}\right\}\right) \\
&\quad = F_{in}(D^2\phi(\mathbf{x}), D\phi(\mathbf{x}), \phi(\mathbf{x}), \mathbf{x})_{i,j,k}^{n+1} + O(h) + O(\xi) \text{ if } \mathbf{x}_{i,j,k}^{n+1} \in \Omega_{in} \backslash \Omega_{Z'} \quad .
\end{aligned}
\tag{A.31}
$$

The rest of the results in equation (A.29) follow using similar arguments. □

Recall the following definitions of upper and lower semi-continuous envelopes

**Definition A.2.** *If $C$ is a closed subset of $\mathbb{R}^N$, and $f(x) : C \to \mathbb{R}$ is a function of $x$ defined in $C$, then the upper semi-continuous envelope $f^*(x)$ and the lower semi-continuous envelope $f_*(x)$ are defined by*

$$
f^*(x) = \limsup_{\substack{y \to x \\ y \in C}} f(y) \quad \text{and} \quad f_*(x) = \liminf_{\substack{y \to x \\ y \in C}} f(y) \quad .
\tag{A.32}
$$

**Lemma A.3** (Consistency)**.** *Assuming all the conditions in Lemma A.2 are satisfied, then the scheme (A.17) is consistent with the HJB equation (4.1), (4.4), (4.5), (4.7), (4.10) in $\Omega$ according to the definition in [7, 5]. That is, for all $\hat{\mathbf{x}} = (\hat{S}, \hat{\mathcal{B}}, \hat{\alpha}, \hat{\tau}) \in \Omega$ and any function $\phi(\mathbf{x}, \psi(\mathbf{x}))$ of the form $\phi(\mathbf{x}, \psi(\mathbf{x}) = \mathcal{B}^2 \psi(z, \alpha, \tau)$, $z = S/\mathcal{B}$, where $\psi$ has bounded derivatives with respect to $(z, \alpha, \tau)$ for $(S, \mathcal{B}, \alpha, \tau) \in \Omega$, and $\mathbf{x}_{i,j,k}^{n+1} = (S_i, \mathcal{B}_j, \alpha_k, \tau^{n+1})$, we have*

$$
\begin{aligned}
&\limsup_{\substack{h \to 0 \\ \mathbf{x}_{i,j,k}^{n+1} \to \hat{\mathbf{x}} \\ \xi \to 0}} \mathcal{G}_{i,j,k}^{n+1}\left(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n+1}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{l,m,p}^{n+1}\right\}_{\substack{l \neq i \\ m \neq j \\ p \neq k}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n}\right\}\right) \\
&\qquad \leq F^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big),
\end{aligned}
\tag{A.33}
$$

*and*

$$
\begin{aligned}
&\liminf_{\substack{h \to 0 \\ \mathbf{x}_{i,j,k}^{n+1} \to \hat{\mathbf{x}} \\ \xi \to 0}} \mathcal{G}_{i,j,k}^{n+1}\left(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n+1}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{l,m,p}^{n+1}\right\}_{\substack{l \neq i \\ m \neq j \\ p \neq k}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)_{i,j,k}^{n}\right\}\right) \\
&\qquad \geq F_*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big).
\end{aligned}
\tag{A.34}
$$

*Proof.* According to the definition of lim inf, there exist sequences $h_q, i_q, j_q, k_q, n_q, \xi_q$ such that

$$h_q \to 0, \quad \xi_q \to 0, \quad \mathbf{x}_q \equiv \left(S_{i_q}, \mathcal{B}_{j_q}, \alpha_{k_q}, \tau^{n_q+1}\right) \to (\hat{S}, \hat{\mathcal{B}}, \hat{\alpha}, \hat{\tau}) \;\; \text{as } q \to \infty, \tag{A.35}$$

and

$$\begin{aligned} &\liminf_{q\to\infty} \mathcal{G}^{n_q+1}_{i_q,j_q,k_q}\left(h_q, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{i_q,j_q,k_q}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{l,m,p}\right\}_{\substack{l\neq i_q\\ m\neq j_q\\ p\neq k_q}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q}_{i_q,j_q,k_q}\right\}\right) \\ &= \liminf_{\substack{h\to 0\\ \mathbf{x}^{n+1}_{i,j,k}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}^{n+1}_{i,j,k}\left(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n+1}_{i,j,k}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n+1}_{l,m,p}\right\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n}_{i,j,k}\right\}\right) \quad . \end{aligned} \tag{A.36}$$

Consider the case where $\hat{\mathbf{x}} \in \Omega_{\alpha_{\min}}$ i.e.

$$\begin{aligned} \hat{\mathbf{x}} &= (S, \mathcal{B}, \alpha_{\min}, \tau) \\ & \quad\; \tau \in (0, T] \;\; ; \;\; S < S_{\max} \quad . \end{aligned} \tag{A.37}$$

Choose $q$ sufficiently large so that

$$0 \le S_{i_q} < S_{\max} \;\; ; \;\; \alpha_{\min} \le \alpha_{k_q} < \alpha_{\max} - v_{\max}(\Delta\tau)_q \quad . \tag{A.38}$$

For $\mathbf{x}_q$ satisfying condition (A.38), and using Lemma A.2, we have

$$\begin{aligned} &\mathcal{G}^{n_q+1}_{i_q,j_q,k_q}\left(h_q, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{i_q,j_q,k_q}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n+1}_{l,m,p}\right\}_{\substack{l\neq i_q\\ m\neq j_q\\ p\neq k_q}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q}_{i_q,j_q,k_q}\right\}\right) \\ &= \begin{cases} F_{in}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) + O(h_q) + O(\xi_q) & \text{if } \mathbf{x}_q \in \Omega_{in}\backslash\Omega_{Z'} \\ F_{\alpha_{\min}}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) + O(h_q) + O(\xi_q) & \text{if } \mathbf{x}_q \in \Omega_{\alpha_{\min}} \\ F_{Z'}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) + O(h_q) + O(\xi_q) & \text{if } \mathbf{x}_q \in \Omega_{Z'} \end{cases} \end{aligned} \tag{A.39}$$

For $\mathbf{x}_q$ satisfying (A.38), since $Z^+ \subseteq Z' \subseteq Z$, it follows from equations (A.19) and (A.26) that

$$\begin{aligned} F_{in}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) &\ge F_{Z'}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) \\ &\ge F_{\alpha_{\min}}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) \quad . \end{aligned} \tag{A.40}$$

We then have

$$\begin{aligned} &\liminf_{q\to\infty} \mathcal{G}^{n_q+1}_{i_q,j_q,k_q}\left(h_q, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{i_q,j_q,k_q}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n+1}_{l,m,p}\right\}_{\substack{l\neq i_q\\ m\neq j_q\\ p\neq k_q}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q}_{i_q,j_q,k_q}\right\}\right) \\ &\ge \liminf_{q\to\infty} F_{\alpha_{\min}}((D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) + \limsup_{q\to\infty}[O(h_q) + O(\xi_q)] \\ &\ge F_*(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}) \quad , \end{aligned} \tag{A.41}$$

where the last step follows since $F_{\alpha_{\min}}, F_{in}$ are continuous functions of their arguments for smooth test functions, and $F_{\alpha_{\min}} \le F_{in}$.

Let $h_q, i_q, j_q, k_q, n_q, \xi_q$ be sequences satisfying (A.35), such that

$$\begin{aligned} &\limsup_{q\to\infty} \mathcal{G}^{n_q+1}_{i_q,j_q,k_q}\left(h_q, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{i_q,j_q,k_q}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{l,m,p}\right\}_{\substack{l\neq i_q\\ m\neq j_q\\ p\neq k_q}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q}_{i_q,j_q,k_q}\right\}\right) \\ &= \limsup_{\substack{h\to 0\\ \mathbf{x}^{n+1}_{i,j,k}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}^{n+1}_{i,j,k}\left(h, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n+1}_{i,j,k}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n+1}_{l,m,p}\right\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \left\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi\big)^{n}_{i,j,k}\right\}\right) \quad . \end{aligned} \tag{A.42}$$

Take $q$ sufficiently large so that condition (A.38) are satisfied. It follows from equations (A.40) that

$$F_{Z'}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) \quad \leq \quad F_{in}(D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) \\ \text{if } \mathbf{x}_q \in \Omega_{Z'} \tag{A.43}$$

hence

$$\begin{aligned} &\limsup_{q\to\infty} \mathcal{G}^{n_q+1}_{i_q,j_q,k_q}\Big(h_q, \phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q+1}_{i_q,j_q,k_q}, \big\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n+1}_{l,m,p}\big\}_{\substack{l\neq i_q\\ m\neq j_q\\ p\neq k_q}}, \big\{\phi\big(\mathbf{x}, \psi(\mathbf{x}) + \xi_q\big)^{n_q}_{i_q,j_q,k_q}\big\}\Big) \\ &\leq \limsup_{q\to\infty} F((D^2\phi(\mathbf{x}_q), D\phi(\mathbf{x}_q), \phi(\mathbf{x}_q), \mathbf{x}_q) + \limsup_{q\to\infty}[O(h_q) + O(\xi_q)] \\ &\leq F^*(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}) \quad . \end{aligned} \tag{A.44}$$

Similar arguments can be used to prove (A.33-A.34) for any $\hat{\mathbf{x}}$ in $\Omega$. □

**Remark A.3** (Need for Definition of Consistency [7])**.** *Note that in view of equation (A.39), there exist points near the boundaries where the discretized equations are never consistent in the classical sense with equations (4.1), (4.4-4.5) and (4.10). Classical consistency would require that $Z' = \emptyset$, which could only be achieved by placing restrictions on the timestep and $(\Delta\alpha)_{\min}$. These artificial restrictions are not required for the more relaxed definition of consistency (A.33-A.34).*

## A.4 Monotonicity

Using the methods in [18] it is straightforward to show show that scheme (A.17) is monotone.

**Lemma A.4.** *If the discretization (5.4) is a positive coefficient discretization, and interpolation scheme (5.15-5.16) is used with linear interpolation in the $S \times \alpha$ plane, then discretization (A.17) satisfies*

$$\begin{aligned} &\mathcal{G}^{n+1}_{i,j,k}\big(h, V^{n+1}_{i,j,k}, \{X^{n+1}_{l,m,p}\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \{X^n_{i,j,k}\}\big) \\ &\leq \mathcal{G}^{n+1}_{i,j,k}\big(h, V^{n+1}_{i,j,k}, \{Y^{n+1}_{l,m,p}\}_{\substack{l\neq i\\ m\neq j\\ p\neq k}}, \{Y^n_{i,j,k}\}\big) \quad ; \quad \textit{for all } X^n_{i,j,k} \geq Y^n_{i,j,k}, \forall i,j,k,n \quad . \end{aligned} \tag{A.45}$$

Note that if the similarity reduction (3.12) is valid, then we can replace $X^n_{i,j,k}$ by $X^n_{m,0,p}, X^n_{m,1,p}$, and $Y^n_{i,j,k}$ by $Y^n_{m,0,p}, Y^n_{m,1,p}$, using equations (5.15-5.16). Hence it follows from Lemma A.4 that the discretization is monotone in terms of $X^n_{m,0,p}, X^n_{m,1,p}$, $\forall m,p,n$. Since $X^n_{m,0,p}, X^n_{m,1,p}$ are essentially the discretized values of $\psi(S/\mathcal{B}, \alpha, \tau)$ in equation (5.18), we have the precise form of monotonicity required in [7].

## A.5 Convergence

We make the assumption that there exists a unique, continuous viscosity solution to equation (3.2) with boundary conditions (4.4-4.5 ), (4.10), (4.7), at least in $\Omega_{in}$. This follows if the equation and boundary conditions satisfy a strong comparison property.

**Assumption A.1.** *If $u$ and $v$ are an upper semi-continuous subsolution and a lower semi-continuous supersolution of the pricing equation (3.2) associated with the boundary conditions (4.4-4.5 ), (4.10), (4.7), then*

$$u \leq v \quad ; \quad (S, \mathcal{B}, \alpha, \tau) \in \Omega_{in}. \tag{A.46}$$

A strong comparison result was proven in [6] for a a general problem similar to equation (3.2). However, we violate some of the assumptions required in [6] (i.e. the domain is not smooth).

We can now state the following result

**Theorem A.1** (Convergence)**.** *Assume that scheme (A.17) satisfies all the conditions required by Lemmas A.1, A.3, A.4, and that Assumption A.1 holds, then scheme (A.17) converges to the unique, continuous viscosity solution to problem (3.2), with boundary conditions (4.4-4.5), (4.10), (4.7), for* $(S, \mathcal{B}, \alpha, \tau) \in \Omega_{in}$.

*Proof.* This follows from the results in [7, 5]. □

**Remark A.4.** *Note that as discussed in [23], at points on the boundary where the PDE degenerates, it is possible that loss of boundary data may occur, and the solution can be discontinuous at these points. Hence, in general, we can only assume that strong comparison holds for points in the interior of the solution domain. In this situation, we should consider the computed solution to be the limit as we approach the boundary points from the interior.*

# B Convergence of the Expected Value

Given the optimal control determined from the solution to equation (5.11), then equation (5.13) is a discretization of the linear PDE (4.11) with a classical solution. The discretization (5.13) is easily seen to be consistent. It is perhaps not immediately obvious that scheme (5.13) is $l_\infty$ stable, in view of the similarity reduction (5.15-5.16), with the control determined from equation (3.2). Note that $|\mathcal{B}^n_{\hat{j}}/\mathcal{B}^*|$ may be greater than unity (see equations (5.15-5.16)). However, we note that

$$\begin{aligned} U^n_{i,j,k} &\simeq E^{t=0}_{v^*}[\mathcal{B}_L] \\ V^n_{i,j,k} &\simeq E^{t=0}_{v^*}[(\mathcal{B}_L)^2] \end{aligned} \tag{B.1}$$

so that if $V^n_{i,j,k}$ is bounded, then

$$Var[\mathcal{B}_L] = E^{t=0}_{v^*}[(\mathcal{B}_L)^2] - (E^{t=0}_{v^*}[\mathcal{B}_L])^2 \geq 0 \quad . \tag{B.2}$$

would imply a bound on $(U^n_{i,j,k})^2$.

Stability in the $l_\infty$ norm for $U^n_{i,j,k}$ is a consequence of the following Lemma.

**Lemma B.1** (Stability of scheme (5.13))**.** *If* $U^{n+1}$ *is given by (5.13), with the discrete optimal control determined by the solution to equation (5.11), a positive coefficient method is used to discretize the operator* $\mathcal{L}$ *as in equation (5.4), the discrete similarity interpolation operators are given by equations (5.15-5.16), with linear interpolation in the* $S \times \alpha$ *plane, and the payoff conditions given by equations (5.12) and (5.14), then*

$$(U^n_{i,j,k})^2 \quad \leq \quad V^n_{i,j,k} \quad ; \quad \forall i, j, k, n \quad . \tag{B.3}$$

*Proof.* Define $V^n_{j,k} = [V^n_{0,j,k}, ..., V_{i_{\max},j,k}]^t$, with $\mathcal{L}_h$ being the $i_{\max} + 1 \times i_{\max} + 1$ matrix defined in equation (5.4). Write equations (5.11) and (5.13) as

$$\begin{aligned} [I - \Delta\tau\mathcal{L}_h]V^{n+1}_{j,k} = V^{n+}_{j,k} \quad &; \quad V^{n+}_{i,j,k} = \min_{v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}} V^n_{\hat{i},\hat{j},\hat{k}} \\ &\quad\;\; (v^*)^{n+1}_{i,j,k} \in \arg\min_{v^{n+1}_{i,j,k} \in Z^{n+1}_{i,j,k}} V^n_{\hat{i},\hat{j},\hat{k}} \\ [I - \Delta\tau\mathcal{L}_h]\, U^{n+1}_{j,k} = U^{n+}_{j,k} \quad &; \quad U^{n+}_{i,j,k} = \left\{ U^n_{\hat{i},\hat{j},\hat{k}} \right\}_{(v^*)^{n+1}_{i,j,k}} \quad . \end{aligned} \tag{B.4}$$

Since $[I - \Delta\tau\mathcal{L}_h]$ is a diagonally dominant $M$ matrix, and $rowsum(\mathcal{L}_h) = 0$, then

$$\begin{aligned} [I - \Delta\tau\mathcal{L}_h]^{-1} &= G \\ \sum_l G_{i,l} = 1 \quad &; \quad 0 \leq G_{i,l} \leq 1 \quad . \end{aligned} \tag{B.5}$$

Assume $(U_{i,j,k}^{n+})^2 \leq V_{i,j,k}^{n+}$, then since (Jenson's inequality)

$$\left( \sum_l G_{i,l} U_{l,j,k}^{n+} \right)^2 \quad \leq \quad \sum_l G_{i,l} (U_{l,j,k}^{n+})^2 \tag{B.6}$$

we have that $(U_{i,j,k}^{n+1})^2 \leq V_{i,j,k}^{n+1}$. Using the interpolation operators (5.15-5.16) and the definitions of $U^{(n+1)+}, V^{(n+1)+}$ we can see that $(U_{i,j,k}^{(n+1)+})^2 \leq V_{i,j,k}^{(n+1)+}$. Finally, we have $(U_{i,j,k}^0)^2 = V_{i,j,k}^0$. □

Since $V^{n+1}$ is $l_\infty$ stable from Lemma A.1, it follows from Lemma B.1 that $U^{n+1}$ is $l_\infty$ stable.

**Remark B.1.** *Note that Lemma B.1 is true (in general) only if $[I - \Delta\tau\mathcal{L}_h]$ is an $M$ matrix, and linear interpolation is used in operators (5.15-5.16).*

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