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An Introduction to Complete Markets

THE PAST TWO DECADES have seen a proliferation of new and often complex financial securities and commodity contracts in the marketplace. Flipping through the financial pages of the newspaper, one finds, for example, that an investor can purchase the right to buy (at a fixed price on a set future date) a futures contract on 10-year U.S. Treasury notes. Alternatively, one reads of a contract that pays off various dollar amounts depending on the level of the mark-yen exchange rate at the end of October 1992 (see shaded insert at right). Are such arcana practically useful? The theory of complete markets—an important element of modern theoretical economics—can provide some insight.[1]

A complete system of markets is one in which there is a market for every good. This simple statement conceals the significance of the concept, however, by failing to specify what is meant by a "good." By carefully defining "good" to include the date and environment in which a commodity is consumed, economists are able to consider consumption, production and investment choices in a multiperiod, uncertain world. Moreover, they can do so using largely the same utility theory originally developed to analyze timeless certainty. In particular, state-preference theory, which was developed to analyze the completeness of a system of markets, is a pow-

Goldman Sachs Starts Selling New Warrants Betting Yen vs. Mark

By a WALL STREET JOURNAL *Staff Reporter*

NEW YORK—Goldman, Sachs & Co. began selling five million warrants at $3.50 each that allow investors to bet that the Japanese yen will strengthen against the German mark over the next two years.

The warrants were designed by Goldman but they are being offered by AT&T Capital Corp., a finance arm of the communications company. Co-managers of the offering are Oppenheimer & Co. and Dean Witter Reynolds Inc.

According to currency option traders these are the first cross-currency warrants to be offered in the U.S., though such warrants have been sold in Europe. The warrants started trading yesterday on the New York Stock Exchange.

Most warrants, like options, give holders the right but not the obligation, to buy or sell securities, currencies and other instruments at a specific price on or before a certain date. This warrant is more complicated because it allows holders to bet on the relative strength of two foreign currencies with a payoff in dollars.

The Wall Street Journal publishes a daily table of cross-rates between various currencies. For example, the closing cross-rate yesterday was 85.696 yen to buy one mark.

The warrants entitle an investor to a payment in U.S. dollars if the number of yen to purchase one mark drops below 85.20. The intrinsic value of the warrant is calculated from a formula specified in the warrant's prospectus. Each warrant initially will represent $50 of currency and expires Oct. 30, 1992.

Holders of at least 1,000 warrants will not have to wait until expiration to exercise them. "American style" warrants can be exercised on any business day. Holders will also be able to sell them on the Big Board to other investors.

[1]To be correct, one should refer to a "complete system of markets." This phrase can be unwieldy at times, however, and it is usually abbreviated to "complete markets" or "complete market." These three terms are used interchangeably here (see glossary).

A Glossary for Complete Markets

Arbitrage: A collection of payoff vectors with zero net cost, whose net payoffs in all outcomes are non-negative and not all zero. In other words, a set of transactions that results in getting something for nothing.

Arrow-Debreu security: Same as *pure security.*

Bet: A contract whose payoffs are a constant amount in all outcomes that are elements of a specified event, and zero in all other outcomes. For example, a contract that pays off $5 in the event "M wins" and zero otherwise is a bet.

Call option: A contract giving the holder the right to buy a particular good at a prespecified future date for a prespecified price (see also *put option*).

Complete market: A system of markets in which every agent is able to exchange every good, either directly or indirectly, with every other agent. Also called a complete system of markets.

Consistent odds: Betting odds on some collection of events that are inversely related to a probability measure over those events, according to the formula: $O_e = (1/P(e)) - 1$, where O_e is the odds on the event e, and P(e) is the probability of e occurring.

Efficient allocation: Same as *Pareto-optimal allocation.*

Event: A collection of outcomes. For example, "M wins" is an event that includes the rankings M-T-C and M-C-T (see also *outcome*).

Futures contract: A standardized contract to purchase a commodity, which is to be delivered at a specific date in the future.

General equilibrium: A feasible combination of prices, consumption choices, and production choices such that there is no excess supply or demand for any good.

Hedge: To arrange a combination of contracts such that low payoffs for one contract in a given state are offset by high payoffs for another contract in the same state. The result is a portfolio with reduced variability of payoffs across states (see also *speculate*).

Incomplete market: A system of markets which is not complete.

Linear combination: A vector produced by adding (or subtracting), element by element, any scalar multiples of two or more other vectors.

Linearly independent: A set of vectors is linearly independent if no vector in the set can be expressed as a linear combination of the other vectors in the set.

Option: See *call option* and *put option.*

Outcome: A complete specification of the realizations of all relevant variables over the entire relevant time horizon (see also *event*).

Pareto-optimal allocation: An allocation of economic resources among individuals such that no individual can be made better off without making some other individual worse off.

Payoff vector: A vector whose elements represent the amount paid of each good in each outcome under a particular contract.

Primitive security: Same as *pure security.*

Pure security: A contract that pays off one unit (usually one dollar) in a particular outcome and nothing in all other outcomes.

Put option: A contract giving the holder the right to sell a particular good at a prespecified future date for a prespecified price (see also *call option*).

Redundant bet: A bet whose payoff vector can be constructed as a linear combination of the payoff vectors of other available bets.

Scalar multiple: A vector produced by scaling another vector up or down, i.e., by multiplying each element of the latter by the same real number.

Short sale: The sale of a borrowed good with the intention of later repurchasing that good at a profit.

Span a space: A collection of vectors is said to span a space if the collection has the property that any point in the space is reached by some linear combination of those vectors.

Speculate: To purchase (or sell) a good, intending to resell (repurchase) later, where the motive is to profit from an anticipated change in price rather than any gain accruing through use of the good (see also *hedge*).

State: Same as *outcome*.

State-claim: A contract that pays off differing amounts (or different goods) under different states of the world.

State-contingent-claim: Same as *state-claim*.

State of the world: Same as *outcome*.

Vector: An ordered collection of numbers. A vector can also be represented as a point or an arrow in space.

erful tool with which to study behavior under uncertainty.

The purpose of this paper is to introduce the non-specialist to the basic theory of complete markets, providing the reader with an insight into the nature of markets and recent financial innovations in particular. The paper first introduces the major concepts of the state-preference approach to uncertainty, illustrating them with a parimutuel gambling example. In this framework, the notion of completeness arises naturally as the extreme case in which bettors have the greatest range of opportunities to bet on the outcome of a race. The terminology is then transferred to an economic context, where again, complete markets provide consumers, producers and investors the most flexibility in allocating payoffs and planning for uncertain contingencies. Particular attention is given to the markets for futures and options.

Such securities are shown to improve the efficiency of the marketplace, a result with implications for regulatory policy.

In the real world, systems of markets are not complete, as we shall see.The notion of completeness, however, is of interest for two reasons. First, it serves as a theoretical benchmark, relative to which incompleteness can be assessed; such a comparison might, for example, suggest whether incompleteness implies inefficiency in a particular model. Second, although the notion of market completeness appears most often in theoretical discussions, the ideas involved can also be applied to more realistic problems. For example, in the state-preference context, markets for so-called "derivative" securities — futures and options — add value by providing investors with flexibility in fashioning their portfolios; thus, they make systems of markets less incomplete. The popularity of such securities can thus be explained from a theoretical perspective that incorporates complete markets. In some cases, the theory can even suggest new markets that would alleviate existing incompleteness.

THE THEORETICAL APPARATUS

The tools and results of the theory of complete markets represent one the most significant developments of theoretical economics in this century.[2] At the same time, the concepts embedded in the theory are very general and have been used in many other economic contexts. Thus, our first task is to explore the basic structure of state-preference theory. We do this with a simple gambling example, because it involves a well-defined and relatively small collection of outcomes and payoffs in an uncertain environment.[3]

[2]The theory can be traced to the work of Arrow (1964), Debreu (1959), Arrow & Debreu (1954) and McKenzie (1954) in the mid-1950s. Both Arrow and Debreu were later awarded the Nobel Memorial Prize in Economics (Arrow in 1972, Debreu in 1983), largely for their work in developing the theory of complete markets and applying it to the problem of general equilibrium. The theory is often cited in the guise of its two most common avatars, the Arrow-Debreu general equilibrium theory and state-preference theory; it also often appears as its implicit counterpart, the theory of incomplete markets. The literature applying state-preference theory and the theory of complete markets is lengthy; for a partial list, consult the references in Radner (1982) and Debreu (1982).

[3]Previous articles have recognized the connection between gambling and economic activity, especially financial markets. See, for example, Gabriel & Marsden (1990) or Asch, Malkiel & Quandt (1984), and the references therein.

State-Claims Defined

One dominant theme of state-preference theory and complete markets is uncertainty. State-preference theory incorporates uncertainty by defining outcomes, or potential future states, only one of which will ultimately be realized. The theory has been fruitfully applied in many areas of economics, especially in the study of financial markets. For now, however, consider an imaginary racetrack called Portfolio Downs.

We are bettors at Portfolio Downs, hoping to win enough for an early retirement. To make our life simpler, the track's management has done away with the tote board.[4] Instead, all bets are placed at fixed odds with the track's official bookmaker. This allows us to confine our uncertainty to the race itself, without worrying that the posted odds might shift after we've placed a bet. Furthermore, to keep the number of contingencies to a minimum, only one race will be run today.

Our first assignment at the track is to define the set of relevant states and payoffs. A *state of the world* is defined as a complete specification of the values of all relevant variables over the entire relevant time horizon. A state of the world is also called an *outcome*, or simply a *state*. In an economic system, the relevant variables might include the structure of the tax code, global weather patterns, infant mortality rates, scientific discoveries, etc. The relevant time horizon might be stated in periods as long as a decade or as short as a second; it might extend into both the past and the future for a few hours or many centuries.

At the racetrack, however, matters are much simpler: a state of the world is a complete listing of the finishing position for every horse in today's race.[5] For example, if there are only three horses in the race:

1. Tricky Bond (T)
2. Mastercharger (M)
3. Charge Me Interest (C)

then there are six possible states of the world, which we can write in win-place-show order: T-M-C, T-C-M, M-T-C, M-C-T, C-T-M, C-M-T.[6] Although we may have definite opinions at the start of the day, we cannot know the state of the world for sure until the race has been run. An *event* is a collection of one or more states. Thus, for example, "Mastercharger wins the race" is an event. It includes two states, M-T-C and M-C-T, both of which are consistent with the stated criterion.

All *bets* are placed on events. If the state of the world that ultimately occurs is an element of the event, then the bet pays off at the fixed odds; otherwise the bet pays nothing. Bets are a special type of *state-contingent claim* (or simply *state claim*). More generally, a state claim is a contract that pays off differing amounts—perhaps even different goods—under different states of the world.

A state claim can be represented as a *payoff vector* with one element for each state of the world. The notion of a payoff vector, central to the theory of complete markets, is stated in terms of the mathematics of vectors, linear algebra.[7] In our example, a $2 bet on Mastercharger to win that pays off at 4-to-1 odds can be represented by the vector: (0,0,$10,$10,0,0), where the positions in the vector are in the same order as the states listed above. Alternatively, a $2 trifecta bet on the ranking C-T-M that pays off at 3-to-1 odds has the payoff vector: (0,0,0,0,$8,0).

In state-preference theory, a market is equivalent to a payoff vector: a market represents the ability to exchange goods or payoffs. At the racetrack, we exchange pre-race dollars for a state-contingent bundle of post-race dollars. Some exchanges are available directly: we can exchange $2 pre-race for the post-race vector (0,0,$10,$10,0,0). Other exchanges can be constructed indirectly: we might exchange $4 pre-race for the post-race vector (0,0,$10,$10, $4,$4) by placing two separate $2 bets.

The key to the theory of complete markets is to deal with such combinations in a systematic fashion. A system of markets is complete when

[4] Some basic racing and betting terminology is defined in the shaded insert on page 36.

[5] Note that if there were more than one race, a state of the world would involve a complete listing of all horses in all races. In other words, a single state of the world describing all the day's races would be ultimately realized.

[6] In general, the number of permutations is given by the factorial function. E.g., 3! = 3•2•1 = 6.

[7] See the shaded insert on pp. 37-38 for a quick introduction to linear algebra.

Parimutuel Betting

The term parimutuel comes from the French words *pari* meaning "stake" and *mutuel* meaning "mutual." It describes the mechanics of determining betting odds and payoffs in the system commonly used at American racetracks. Although Portfolio Downs uses a bookmaker rather than a parimutuel system, the terminology is the same with both methods. In a parimutuel system, bets are grouped together in pools corresponding to the type of bet (e. g., win, place, show, quinella, etc.). Each pool is treated separately. The track takes a cut off the top (the take), and the remaining pool is pro-rated among the winning bettors according to the amount bet.

Odds are stated as the ratio of profit to wager; for example, a winning $2 bet at 3-to-1 odds pays back the $2 initial wager plus $3 for each dollar bet, making a total of $8. The odds themselves are a function of the amount bet on each event in the pool. In particular, ignoring the take, the odds on a horse are given by: $O_h = (B/b_h) - 1$, where O_h is the odds on horse h in the pool, B is the total size of the pool, and b_h is the amount in the pool that was bet on horse h. Odds are posted on the tote board and fluctuate according to the relative volume of bets placed up until the time of the race. Implicit in this arrangement is a sort of probability measure representing the aggregate beliefs of the bettors. Note that $P_h \equiv 1/(1+O_h) = b_h/B$. This quantity can be treated as the implicit probability that a bet on horse h will pay off. The parimutuel system can have some curious properties, however. For example, if many people bet a speedy horse to win, ignoring the place pool, the final odds on that horse to place can exceed the odds on the same horse to win, even though the place bet is safer than the win bet. Although this clearly implies inconsistent odds, it can and occasionally does occur.

Bettor's Glossary

Daily Double: A bet that specifies the two winning horses in two designated races; i. e., both horses must win for the bet to pay off.

Exacta: Same as a *perfecta*.

Odds: The ratio of net profits to amount wagered for a winning ticket.

Parlay wager: A sequence of two or more bets in which all contingencies are chosen in advance, and in which the full proceeds of earlier bets are wagered on the next bet in the sequence. For example, a daily double bet is a parlay wager.

Perfecta: A bet that specifies the top two horses, in order (see also *quinella*).

Place: To come in second in a race. Also, a bet that a specified horse will at least place (i. e., win or place).

Quinella (or quiniela): A bet that specifies the top two horses, but does not specify their order (see also *perfecta*).

Show: To come in third in a race. Also, a bet that a specified horse will at least show (i. e., win, place or show).

Take: The cut from the betting pool that is taken by the racetrack to cover taxes, overhead, etc.

Totalizator or tote board: A system for reporting a horse's current betting odds implicit in the mutuel pool.

Trifecta: A bet that specifies the top three finishers, in order.

Win: To come in first in a race. Also, a bet that a specified horse will win.

we can arrange a portfolio with any conceivable payoff vector. We may not want certain payoff vectors, and even among those we do find desirable, our decision on which portfolio ultimately to arrange will depend both on their prices and our resources. These issues of desirability and affordability are secondary for the notion of completeness, however. The important characteristic of a complete system of markets is that, without a wealth constraint, any conceivable payoff vector can be arranged. In terms of linear algebra, a complete system of markets is one in which the set of available bets contains enough linearly independent payoff vectors to span the space of all conceivable payoff vectors.

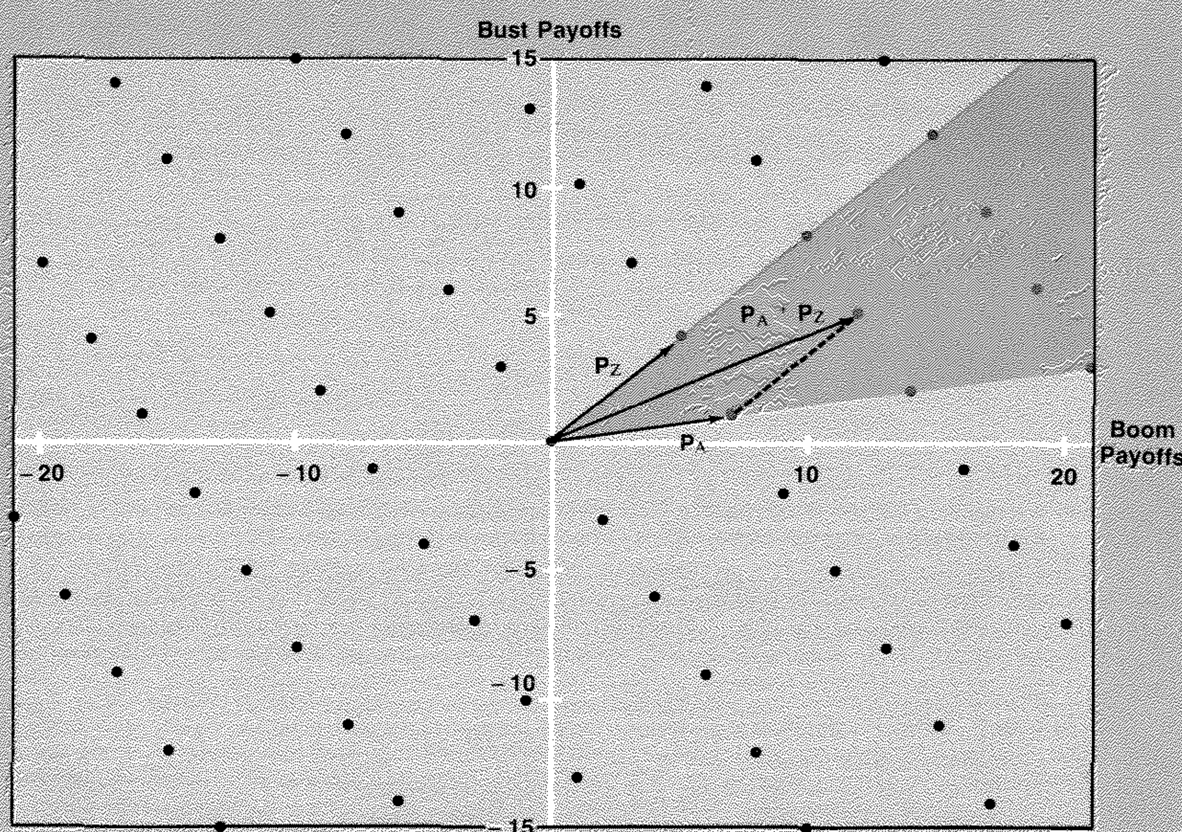
The Algebra of Payoff Vectors

The mathematical concept of a spanned vector space and the economic concept of a complete system of markets are closely related. Although the algebraic logic remains the same for any number of states, we can gain some intuition for the problem by considering the special case in which there are only two states of the world. For this case we can graph the relationships involved. For example, let the two states be boom and bust, and consider two securities in which we might invest, Abelard Abrasives and Zwingli Swings. We don't know yet whether the economy will go boom or bust, but we do know how the securities will perform in each case. Security performance is represented by the payoff vectors: $P_A = (7,1)$, and $P_Z = (5,4)$. This situation is depicted in figure 1, where each state is assigned to an axis, and the payoff vectors are represented by arrows extending from the origin.

We can add vectors by simply adding together the corresponding components. Thus, if we buy a portfolio of one share of each security, our payoff vector will be $P_{A+Z} = P_A + P_Z = (7+5, 1+4) = (12,5)$. Graphically, this is achieved by connecting the tail of one vector to the head of the other; the resulting vector is a new arrow extending from the origin to the location of the unconnected arrowhead. It does not matter which head is attached to which tail, because $(7+5, 1+4) = (5+7, 4+1)$. A little experimentation reveals that buying portfolios of different numbers of the two securities results in payoff vectors to the purple points in the graph.

We can also calculate multiples of a vector in an analogous fashion. For example, $P_Z + P_Z = 2P_Z$, which is obtained by multiplying each element of the vector by two. More generally, we can multiply a vector by any number, simply by performing the multiplication

Security Payoff Vectors



on each of the elements.[1] Graphically, this results in the lengthening or shortening of the original vector, without changing its direction. In our securities example, if we are allowed to purchase fractional amounts of the securities for our portfolio, then we can obtain a payoff vector that lies anywhere in the purple-shaded wedge.

Scalar multiplication of a vector also works with negative numbers. If we multiply a vector by a negative number, the resulting arrow in the graph lies along the same line as the original, but points in the opposite direction. For our securities example, this is the equivalent of a *short* sale, the sale of a borrowed security. Short sales are generally made with the intention of repurchasing the security later at a (hopefully) lower price. If we sell a security short, we are responsible for reimbursing the lender the amount of the security's payoff when the state of the world is revealed — hence the negative payoffs. Mathematically, this is the same as adding a negative security to our portfolio: $P_Z - P_A = P_Z + (-P_A)$. Graphically, we add the arrows in the same way, making sure that we use the $(-P_A)$ vector in the addition. If we are allowed to sell shares of Abelard and Zwingli short, we can add the black points to our list of potential portfolio payoffs.

Finally, if we are also allowed to sell fractional amounts short, we can add the entire grey shaded region to our feasible set. We can arrange any payoff combination in the two-dimensional plane. In this case, when any sort of addition and scalar multiplication is allowed, we say we can use all *linear combinations* of the securities. The payoff vector for a linear combination of the securities will always take the form: $P_p = x_a P_A + x_z P_Z$, where x_a and x_z are real numbers. If there are no restrictions on x_a and x_z, then *all* linear combinations are allowed.

In our example, the payoff vectors are chosen so they do not lie along the same line. To describe this non-parallel quality, we say the two vectors are *linearly independent*. If they did lie along the same line, say instead that $P_Z = 2P_A = (14,2)$, then the vectors would be *linearly dependent*. In the case of linearly dependent vectors, we cannot arrange portfolios with any payoff combination in the plane; it's as if we only had a single security to choose from. When the two vectors are linearly independent, so that any payoff combination is accessible, the two vectors are said to *span* the plane.

All of this generalizes directly to situations with more than two states of the world. The only difference is that it is more difficult to graph three states of the world, and impossible to graph four or more. Whatever the number of dimensions, a collection of vectors always spans something; the question of interest for us is what that something is. For example, two vectors in a three-dimensional space can span at worst a single point — the origin (let $P_Z = P_A = (0,0)$). At most they span a two-dimensional plane. In either case, there remains an infinity of points in the three-dimensional space that cannot be reached by any combinations of the two vectors.

[1]This is also called "scalar multiplication," to distinguish it from "vector multiplication," in which two vectors are multiplied.

An Incomplete Market

To illustrate why and how completeness might be valuable to us, let's impose some more conditions on Portfolio Downs. This first example illustrates both an incomplete system of markets and a redundant bet. A bet is *redundant* if its payoffs in every outcome can be duplicated by buying or selling some combination of the other available bets. The system of markets is *incomplete* if the number of outcomes exceeds the number of non-redundant bets. Later on, we shall replace one of these redundant bets with a non-redundant bet, thus completing the market.[8]

[8]Readers familiar with linear algebra should note that *redundancy* is defined as the linear dependence of the set of payoff vectors. Roughly speaking, if the market is incomplete, our payoff vectors are restricted to a "flatland" within the larger space of possible state claims.

Table 1

BET	ODDS	PAYMENT	PAYOFFS BY OUTCOME					
			T-M-C	T-C-M	M-T-C	M-C-T	C-T-M	C-M-T
T wins	7-3	−$2	$6.67	$6.67	0	0	0	0
T places	2-3	−$2	$3.33	$3.33	$3.33	0	$3.33	0
M wins	4-1	−$2	0	0	$10.00	$10.00	0	0
M places	9-11	−$2	$3.64	0	$3.64	$3.64	0	$3.64
C wins	1-1	−$2	0	0	0	0	$4.00	$4.00
C places	3-17	−$2	0	$2.35	0	$2.35	$2.35	$2.35

Suppose that the race is the three-horse affair described earlier. To make things challenging, further suppose that the resident bookmaker will accept only win bets and place bets, refusing to take the seemingly more complicated quinellas, trifectas, etc.[9] Finally, to make completeness an intriguing proposition, suppose that we get a hot tip from our pal in the stables that the race is fixed: the final outcome will be M-T-C. Because we trust our pal, we now want to bet only on this single state of the world. The key question for market completeness is: can we do it, using only win and place bets?

First, note that we aren't satisfied with a bet on Mastercharger to win: in this bet, we would also be buying a payoff in state M-C-T, which we're convinced won't occur. With consistent odds, our payoff per dollar wagered can't be as high betting Mastercharger to win (i.e., betting on the event, [M-T-C, M-C-T]), or betting Tricky Bond to place (i.e., the event, [T-M-C, T-C-M, M-T-C, C-T-M]), as it would be on the M-T-C trifecta bet.[10] We want a portfolio of bets with a payoff vector that looks like this: (0,0,x,0,0,0), where x is some positive number, to make the maximum profit on our bet. The number x should equal exactly the payoff we would get on the M-T-C trifecta bet, if only the bookmaker would allow trifecta bets.

It turns out we cannot achieve this payoff vector from the available bets, an upshot of the fact that this system of markets is not complete. In practical terms, if the system of markets were complete, then we could get higher odds on a bet paying off in the event M-T-C than we can get with the current incomplete set, as will be demonstrated below.

To see more clearly what this means, let's experiment with the numbers. Suppose that the bookmaker gives odds on the six allowable bets, implying the six payoff vectors listed in table 1. These payoffs have been rounded off to the nearest penny. They include the initial wager of $2 plus any profit from the wager, which depends on the odds.[11]

To say that one of the bets is redundant means that its payoffs in the six outcomes can be replicated by an appropriate portfolio of the other bets.[12] The number of outcomes (six) equals the number of bets (six), but exceeds the number of non-redundant bets (five); the system of markets is thus incomplete. Any payoff vector attainable by combinations of the six bets is still attainable if we disallow one of the redundant bets and include combinations consisting only of the other five.

[9] Show bets are superfluous here, because in a three-horse race, all show bets automatically pay off. Thus, they're not really wagers at all. Also note that, although a trifecta might seem complex, it is in a sense the simplest bet here, because it pays off in only one state. This would not be true of trifecta bets if there were more than three horses or more than one race.

[10] "Consistent" simply means that the odds on any event are inversely related to the probability, in the bookmaker's view, of it occurring — the more likely an event, the lower its odds. Consistent odds are related to the probability of an event, e, by the formula: $O_e = (1/P(e)) - 1$, where O_e is the odds ratio for the event e, and P(e) is the probability that e will occur. With consistency, the probabilities of all the individual outcomes add up to 100 percent.

[11] To convert between odds and payoffs, use the formula: $p = b(1 + O_e)$, where p is the payoff, b is the size of the bet (e.g., $2), and O_e is the odds ratio (e.g., 7-3 odds imply $O_e = 7/3 \approx 2.33$).

[12] Redundancy requires that we be able to take bets as well as place them. By taking bets, we can arrange for a negative payoff in certain events. For convenience, we assume that the bookmaker at Portfolio Downs meets this need by placing bets at his posted odds. It is also convenient if we assume that bets can be both taken and placed in any fractional amount, allowing us to fine-tune our portfolio.

Table 2

BET	ODDS	PAYMENT	PAYOFFS BY OUTCOME T-M-C	T-C-M	M-T-C	M-C-T	C-T-M	C-M-T
T wins	7-3	−$24	$80.00	$80.00	0	0	0	0
T places	2-3	$24	−$40.00	−$40.00	−$40.00	0	−$40.00	0
M wins	4-1	−$16	0	0	$80.00	$80.00	0	0
M places	9-11	$22	−$40.00	0	−$40.00	−$40.00	0	−$40.00
C wins	1-1	−$40	0	0	0	0	$80.00	$80.00
Total:		**−$34**	**0**	**$40.00**	**0**	**$40.00**	**$40.00**	**$40.00**

For example, we can construct the sixth bet as a portfolio of the other five. Consider first a $34 bet on Charge Me Interest to place.[13] Ignoring the rounding error, 17($2.35) = $40.00, and our payoffs would be (0,$40,0,$40,$40,$40) under the six possible outcomes. Now consider a portfolio that consists of different amounts of the other five bets, either taken (negative payoffs) or placed (positive payoffs). In particular, let the amounts be those listed in table 2. For example, we take 12 $2 bets on Tricky Bond to place, while placing 12 $2 bets on the same horse to win. The payoffs to our portfolio are given by totaling the six columns in the table. The portfolio yields the same result as if we had placed a $34 bet on Charge Me Interest to place. In fact, since we placed $80 in bets while taking $46 worth, our net investment in the portfolio is $(−80+46) = −$34. The fact that equivalent payoff vectors require the same investment reveals that our bookmaker has set the odds (i.e., payoffs) in a consistent way.

One way to look at the redundancy of one of our bets is that we cannot synthesize the trifecta bet that we're after. Even though we would place such a bet at any consistent positive odds (because we're convinced we know the outcome) that bet is neither offered directly nor can we synthesize it from the others. In other words, we can't get there from here.

Completing The Market

Another way to look at this is that the bookmaker can drop a redundant bet from the list of allowable bets without affecting our opportunities. It turns out that, in our example, the six bets are mutually redundant: any bet that is omitted can be reconstituted from the remaining five. The same procedure that was just used to reconstitute the sixth bet could be applied to generate any of the individual bets from the other five.[14] Let's drop a bet then and replace it with a non-redundant bet.

For example, suppose the bookmaker does not offer a bet on Mastercharger to place. Instead of that bet, he gives 3-1 odds on a trifecta bet on the ranking C-M-T. The odds and the implicit payoffs now available to us are given in table 3. The fourth row of the original payoff array has been replaced by the payoffs to the new trifecta bet. The question is still whether we can arrange a portfolio of bets that pays off only when the ranking is M-T-C. That is, can we synthesize an M-T-C trifecta bet? More generally, is the system of markets complete? The answer to both questions is yes. The system of markets is complete, which implies that we can achieve any payoff vector, including the payoff vector that corresponds to an M-T-C trifecta bet.

To synthesize that bet, combine the bets as described in table 4. The result is a net two-dollar investment that only pays off if the final outcome of the race is M-T-C. The payoff in that case is $40, implying 19-to-1 odds.[15] By creating a bet that pays off under such narrow circumstances, we have maximized the return on our $2 investment (assuming our pal in the stables is trustworthy!). With a complete system of markets, of course, we can also generate a safer portfolio—that is, one that pays off in

[13]Using 17 two-dollar tickets ($34) simply keeps all the following payoffs in terms of nice round numbers. We could just as well scale all the amounts and payoffs down by a factor of 17 to show the same result.

[14]This is not necessarily the case with redundant bets. Consider, for example, three redundant bets on two outcomes: (0,1), (1,1) and (2,2). If (2,2) is dropped, it can be reconstituted from (1,1). If (0,1) is dropped, however, it cannot be synthesized from (1,1) and (2,2).

[15]Again, the scale of the payoffs is unimportant here. Forty dollar amounts are used to keep the numbers consistent with the previous example. The important result is that the portfolio pays off only in one specific state of the world.

Table 3

BET	ODDS	PAYMENT	T-M-C	T-C-M	M-T-C	M-C-T	C-T-M	C-M-T
			PAYOFFS BY OUTCOME					
T wins	7-3	−$2	$6.67	$6.67	0	0	0	0
T places	2-3	−$2	$3.33	$3.33	$3.33	0	$3.33	0
M wins	4-1	−$2	0	0	$10.00	$10.00	0	0
C-M-T trifecta	3-1	−$2	0	0	0	0	0	$8.00
C wins	1-1	−$2	0	0	0	0	$4.00	$4.00
C places:	3-17	−$2	0	$2.35	0	$2.35	$2.35	$2.35

Table 4

BET	ODDS	PAYMENT	T-M-C	T-C-M	M-T-C	M-C-T	C-T-M	C-M-T
			PAYOFFS BY OUTCOME					
T wins	7-3	$12	−$40.00	−$40.00	0	0	0	0
T places	2-3	−$24	$40.00	$40.00	$40.00	0	$40.00	0
M wins	4-1	$0	0	0	0	0	0	0
C-M-T trifecta	3-1	−$10	0	0	0	0	0	$40.00
C wins	1-1	$20	0	0	0	0	−$40.00	−$40.00
C places	3-17	$0	0	0	0	0	0	0
Total:		**−$2**	**0**	**0**	**$40.00**	**0**	**0**	**0**

more states—but this would be more costly. For example, to get the same payoff in one additional outcome, M-C-T (i.e., to achieve the payoff vector (0,0,$40,$40,0,0)), would require an $8 bet on Mastercharger to win, a four-fold increase from the investment required for the trifecta bet.

It is no accident that the number of bets in the portfolio equals the number of possible outcomes.[16] Every portfolio we construct is a system of six linear equations in some unknowns, namely the amounts to put into each of the allowable bets. In other words, we start with six state-dependent payoffs (the desired payoff vector), and we seek a combination of weights (i.e., investment amounts) for the available bets that yield those six payoffs. To ensure that such a combination exists, we need at least six unknowns (i.e., available bets) to work with. Furthermore, some collection of six of those available bets must have payoff vectors that are linearly independent.

What if there are more unknowns than equations (i.e., more available bets than elements in the desired payoff vector)?[17] In that case, at least some of the bets must be redundant. Thus, in determining whether the system of markets is complete, it is not safe simply to count equations and unknowns; we must find the largest collection of non-redundant bets. In our example, if we can find six non-redundant bets, then every payoff vector is the result of a unique combination of these six bets. Thus, in our example, the only way to achieve the payoff (0,0,$40,0,0,0) is to combine the bets in the amounts ($12, −$24,0, −$10,$20,0).

Finally, there is one last bit of terminology which appears frequently in state-preference theory. In our example, a trifecta bet has a positive payoff in one state only; in all other states, its payoff is zero. Notice that all other bets consist of various collections of trifecta bets. Clearly then, a set of six different trifecta bets would produce a complete system of markets. This

[16]An allowable bet is included in the count, even if the amount wagered on that bet is zero. The decision to wager nothing on a particular allowable bet is an implicit portfolio decision.

[17]We shall see below that this situation is not usually a practical consideration. The normal problem is that there are not enough unknowns (available bets) rather than too many.

leads to the notion of a reference bet or *pure security*. A pure security is a normalized bet that pays off in only one state. By normalized, we mean that the payoff in the selected state is one unit.[18] To get a normalized payoff, we must adjust the amount invested in that bet. The wager amount that implies the normalized payoff is called the *price of the pure security*. Any state-contingent claim can be regarded as a collection of pure securities. A system of markets is complete if and only if the number of attainable pure securities (either directly or through combination of other securities) equals the number of outcomes.

SOME ECONOMIC APPLICATIONS

Historically, the theoretical economics literature has generally followed two distinct, but entirely compatible lines in interpreting and applying the theory of complete markets. First, led by the initiators of the theory, there were applications to the problem of general equilibrium. Most work in this area now starts with the notion that markets are not complete and proceeds to analyze the nature of equilibrium (or disequilibrium) with incomplete markets.[19]

The other line of research has focused on the relative efficiency of financial markets at allocating risk by providing greater investment flexibility to investors, in the same way that a complete system of markets makes bettors better off at Portfolio Downs. In practice, the general equilibrium applications tend to de-emphasize uncertainty and concentrate on production, consumption and intertemporal optimization. Conversely, the financial market applications tend to ignore real resource constraints and temporal factors, in order to concentrate on uncertainty.

The first section of this paper considered the properties of betting odds at an imaginary racetrack. Our goal in that example was to find some conditions that would ensure complete investment flexibility. In the process, we developed a simple context in which to introduce the terminology of complete markets. An analogy can be made between Portfolio Downs and real markets. Bookmakers manage their risk by laying off bets, while investors manage their risk by hedging their portfolios; bettors are unsure whether Tricky Bond will win the race, while investors are unsure if pet rocks will be popular with consumers; etc.[20] Our next task is to flesh out this analogy to see what importance state-preference theory can have for more general economic analysis. This is done with a series of examples.

Multiple Periods, Multiple Commodities

One of the primary insights of state-preference theory is that the traditional notion of what constitutes an economic good can be readily extended in a way that allows us to examine economic behavior across time and under uncertainty. At the same time, this extension forces us to think of "goods" in a very different way. Our notion of a good is broadened in two directions: time and state.

Examining the time dimension first, consider a simple example involving a banana and an apple. A textbook would tell us that each consumer has a set of preferences such that she either prefers one to the other or is indifferent between them. In our new way of looking at things, however, we must include a time dimension in a description of the commodity. From this new perspective, "apple" is not sufficient to describe a good; one must specify either "apple today" or "apple tomorrow." Merely specifying "apple" is analogous to stating "red" without specifying "red tricycle" or "red Ferrari."

To make the example more specific, suppose that our consumer generally prefers apples at time t, A_t, to bananas, B_t, but that variety is also valuable to her. In this case, the following preferences for consumption over two consecutive days might hold:

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[18]At Portfolio Downs everything is measured in dollars, so a pure security would be a trifecta bet that paid $1 if the outcome was the ranking specified in the bet. Pure securities are also called *Arrow-Debreu securities* or *primitive securities*.

[19]See Geanakoplos (1990) for some examples.

[20]The terms "laying off bets," "covering a position," and "hedging or reinsuring a portfolio" all refer to the same basic process of reversing a transaction with one party by making a countermanding transaction with a third party. For example, a bookmaker who takes a large bet on some event from a bettor can lay it off by placing a bet on the same event with another bookmaker.

Ranking	Bundle	Sequence of fruits
1	(1,0,0,1)	apple today, banana tomorrow
2	(0,1,1,0)	banana today, apple tomorrow
3	(1,1,0,0)	apple today, apple tomorrow
4	(0,0,1,1)	banana today, banana tomorrow,

where the bundles (i.e., payoff vectors) are of the form: (A_0,A_1,B_0,B_1). With our new definition of commodities in mind, the four sequences ranked above can be treated under standard utility theory exactly as four commodity bundles, each composed of a pair of the four different commodities. One might consider, for example, the indifference curves between apples today and bananas today, or budgeting between apples today and apples tomorrow.

An infinite number of other bundles could also be described and ranked, of course. For example, (15,0,3,7) is a possible bundle, one that would be preferable to any of those listed above. Restricting the time dimension to two days and considering the resulting four goods, we see that the system of markets defined by the four bundles here is incomplete: we can get bundle 1 by buying bundles 3 and 4 while selling bundle 2. Thus, there are at most three non-redundant markets for four goods.[21]

Uncertainty

The same approach is used to incorporate uncertainty. To do so, the definition of a commodity is expanded to include the state of the world. Thus, for example, an umbrella in the rain is now a fundamentally different commodity from an umbrella on a sunny day.[22] Similarly, "bananas in peacetime" are different from "bananas in wartime." This new application, however, fundamentally expands our notion of a good. Because states of the world, by definition, are mutually exclusive, we must separate the notion of economic consumption from that of physical consumption. For example, if we purchase the bundle of goods consisting of "an apple in peacetime tomorrow" and "a banana in wartime tomorrow," we shall not be eating both apple and banana tomorrow; we shall eat only one or the other.

Because the additional consideration of time and uncertainty forces such a radical shift in our conception of commodities, it is worthwhile to consider it further. The distinctions between apples and bananas, today and tomorrow, and peace and war serve as a simple basis for an example. Suppose that, today, the political situation is peaceful, but tomorrow's situation is uncertain. This implies two possible states of the world: peace today, peace tomorrow, and peace today, war tomorrow. We thus have six commodities, abbreviated as follows:[23]

Commodity	Abbreviation
apple (peace) today	A – • – 0
banana (peace) today	B – • – 0
apple peace tomorrow	A – P – 1
apple war tomorrow	A – W – 1
banana peace tomorrow	B – P – 1
banana war tomorrow	B – W – 1

Suppose now that the local wholesaler, Whimsical Fruits, sells the "fruit baskets" or state claims described in table 5.

This admittedly contrived example (note, for example, that basket No. 4 consists of 364 apples today and 364 apples tomorrow, plus 364 "bonus" bananas to be delivered only in case of war tomorrow) allows for ready interpretation, because of its similarity to an earlier example. The array of goods here is essentially identical to the payoff array for Portfolio Downs; only the labels and the scale have been changed. At Portfolio Downs, the only distinction between goods was the state of the world; the physical commodity in all cases was cash, and time differences did not exist. At Whimsical Fruits, on the other hand, the goods have different interpretations, and all the amounts are scaled up by a factor of 100; otherwise, the payoff array is identical to that in table 1. We can therefore conclude immediately that the fruit market here is incomplete. For example, a buyer wanting only to buy apples to be delivered tomorrow in

[21]There is, of course, no reason to limit the time dimension to two days. By extending the time dimension indefinitely, we get an infinite number of goods, which would require an infinite number of markets for a complete system.

[22]Street vendors in New York, for example, charge $3 for an umbrella under clear skies, and $5 for an umbrella when it's raining.

[23]Note that there are only two states of the world, distinguished by the political situation tomorrow. Thus, for the commodities dated today, the true state of the world is uncertain, although the current political situation is known to be peace. In principle, therefore, we should distinguish between apple-peace tomorrow-today and apple-war tomorrow-today. As a practical matter, however, we cannot observe this distinction until it is too late for it to affect our behavior. For convenience, the distinction is not made in the example.

Table 5

BASKET	PAYMENT	A-•-O	B-•-O	A-P-1	A-W-1	B-P-1	B-W-1
		NUMBER OF FRUIT IN EACH BASKET					
1	−$200	667	667	0	0	0	0
2	−$200	333	333	333	0	333	0
3	−$200	0	0	1000	1000	0	0
4	−$200	364	0	364	364	0	364
5	−$200	0	0	0	0	400	400
6	−$200	0	235	0	235	235	235

case of peace cannot arrange such a transaction as a mixture of the baskets offered at Whimsical Fruits.

Do Complete Markets Really Exist?

We now have a sufficient context to ask whether markets are complete in reality. The answer from economists who have considered this question is a resounding "No."[24] The problem is the huge number of markets that would be required for completeness to hold. Even with the roughest distinctions (e.g., measuring time in one-year intervals, considering all automobiles in a given year as perfect substitutes, etc.), an astonishing number of states of the world must still be considered. For example, every conceivable future invention must be included. Furthermore, the timing of invention is significant; if a cure for cancer comes in 1997 instead of 1998, that implies a different state of the world. Some would have us distinguish goods by geographical location and even by the identity of the final consumer![25] We need to multiply the number of states by the number of periods and then by the number of physical goods and services. Finally, we need a market with which to exchange every one of these many goods with every other one.

The absence of many markets from a real economy may be explained by transaction costs. For example, a contract for "an apple dated 2006 if a cure for cancer is discovered in 1997" is too costly to arrange, relative to the marginal benefit of such a transaction. Such a commodity is too narrowly defined to be of interest.

Turning to a more plausible example, the International Monetary Market (IMM) of the Chicago Mercantile Exchange (CME) established an organized marketplace for several commodity futures contracts in 1973. Among these were contracts for (1) bagged silver Canadian coins worth C$ 5,000 at face value, and (2) 12,500,000 Japanese yen.[26] While futures trading in silver coins was discontinued several months later after trading volume dwindled, yen futures still trade today with thousands of contracts changing hands every business day.

In hindsight, it seems clear that the convenience provided by a market for bagged Canadian silver coins was outweighed by transaction costs of some form.[27] On the other hand, the successful introduction of a market for yen futures suggests that investors would have faced significant portfolio constraints in their absence. The contrast between these two raises an interesting question: are the incomplete systems of markets observed in the real world efficient in the sense that the missing markets are absent because they are not operationally cost-effective given current trading technology? Or, are they incomplete simply because no one has thought

[24]Radner (1982), p. 930, for example, states that "... it clearly requires that the economic agents possess capabilities of imagination and calculation that exceed reality by many orders of magnitude." Geanakoplos (1990), p. 2, states that, "There is little doubt that permitting the incompleteness of asset markets is a step in the direction of realism."

[25]See, for example, Geanakoplos (1987), p. 116. This last distinction is made in analyzing public goods and externalities.

[26]See Chicago Mercantile Exchange (1974, 1989). All futures contracts specify a future time and place for delivery, in addition to other standardizations. The state-contingent nature of markets for futures and options is considered in greater detail in later examples.

[27]One possible explanation is that the payoffs to coin futures lie in the space spanned by linear combinations of Canadian dollar futures and silver futures, making the coin futures redundant.

to provide the services required to match buyers and sellers?

A full answer, unfortunately, is beyond the scope of this paper, although we return briefly to the issue below when considering futures markets. It is not a simple task to identify beforehand which missing markets impose the most significant constraints by their absence, and thus would be most likely to succeed with investors. The theory of complete markets does tell us, however, that, contingent on the level of transaction costs, identifying and providing such markets can make everyone involved better off. Not surprisingly then, much of the theoretical work in this area investigates the properties of economies where markets are incomplete. Some of the issues involved are presented in the following examples.

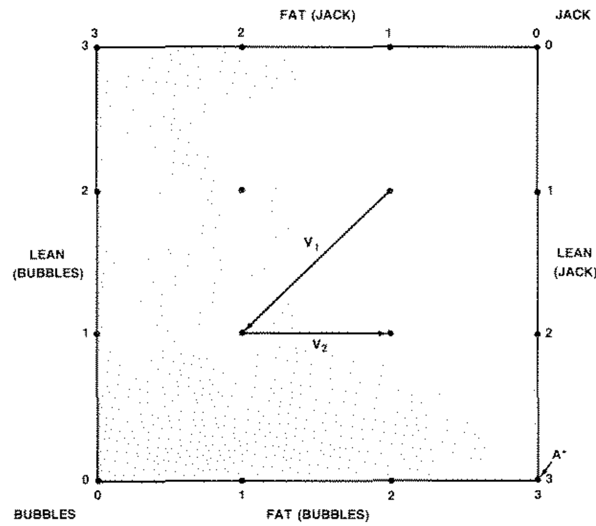
Complete Markets and Efficient Allocations

We first examine one of the most basic problems in economics: how to arrange for the best allocation of available resources. Consider the nursery rhyme of Jack Sprat, but suppose that Jack and his wife, Bubbles, are now divorced and living separately.[28] For simplicity, let's ignore uncertainty and the time dimension. The local market has a special on pork chops, which consist of one fat portion and one lean portion, for \$P per pork chop.[29] This can be represented as a single payoff vector:

	COMMODITY	
PRICE	FAT	LEAN
\$P	1	1

Now consider the allocation of a fixed set of pork chops, say three chops, between Jack and Bubbles. We represent this situation by the modified Edgeworth box in figure 1. Given the special preferences of Jack and Bubbles, there is a unique optimal allocation in this example, represented by the point A^* in the lower-right-hand corner of the box. This is the point at which Jack gets all three lean portions of pork,

Figure 1
The Allocation of Pork



and Bubbles gets all three fat portions.[30] Despite their differences, Jack and Bubbles recognize the value of trade. For example, the vector V_1 in the figure represents a sale of one pork chop from Bubbles to Jack, where Jack owns one pork chop before the sale and two pork chops afterward.

Because there are two commodities but only one transaction (i.e., one payoff vector) in this example, the set of markets is incomplete. The single available market spans the 45-degree line in the payoff space of both consumers; each consumer can achieve any payoff where the number of fat and lean portions is given by some multiple of (1,1). Unfortunately, the optimal point, A^*, does not lie in the space spanned by this available market. The upshot is that the system of markets is incomplete, implying a suboptimal allocation of resources. In other words, the allocation of the six (fat and lean) portions here ideally would be represented by A^*, the unique Pareto-optimal allocation. The available market does not allow them to trade to this

[28]Recall that: "Jack Sprat could eat no fat / His wife could eat no lean / And so betwixt them both / They licked the platter clean."

[29]Note: "To market, to market, to buy a fat pig,/Home again, home again, jiggety-jig."

[30]The optimal allocation is called Pareto-efficient by economists. With a Pareto-efficient allocation, no one can be made better off without harming someone else. In this particular example, there is only one Pareto-efficient allocation, because of the extreme nature of the preferences. In a less extreme case, there would be many Pareto-efficient allocations, depending on preferences, initial endowments and relative prices.

allocation, however, and Jack's fat goes to waste, as does Bubbles' lean. Now, let's complete the system of markets by adding a second allowable transaction: say that Jack and Bubbles agree to trim the fat from the chops and sell it separately to one another at a price of $P/2 per portion of fat. Thus the set of payoff vectors becomes:

PRICE	COMMODITY: FAT	COMMODITY: LEAN
$P	1	1
$(P/2)	1	0

The vector V_2 in the figure represents the sale of a single fat portion from Jack back to Bubbles. By thus completing the market, we have made the optimal allocation attainable.[31]

This same principle of allocational efficiency can be seen at work in more general settings. The major contribution of Arrow and Debreu was to show that an efficient allocation of all commodities is feasible for an economy with complete markets, even with many periods, many physical goods and services, and uncertainty about the future.[32] We next illustrate the importance of complete markets in this context, without considering a full general equilibrium model.

Futures Markets and Risk-Shifting

Uncertainty is a salient concern in many markets. It is common for contracts to require alternative payoffs in different states. Commodity futures contracts, for example, specify the exact physical characteristics of the commodity, the date of delivery and the location of delivery; moreover, they typically provide that "If delivery or acceptance or any precondition or requirement of either is prevented by a strike, fire, accident, action of government or act of God," the directors of the exchange will decide the duties of buyer and seller.[33]

From a practical perspective, however, the implicit state-dependent nature of such contracts is much more important than such explicit stipulations. Commodity futures contracts enable owners of a physical commodity to hedge the value of their inventory exactly against uncertain future fluctuations in price. The state-preference approach can provide useful insights into the nature of such markets.

Futures markets allow people to contract today for future delivery of a specific commodity at a specific price. To see why such a contract is valuable, consider a hypothetical cotton market without a futures market. In particular, suppose that the current price of cotton is 71 cents per pound ($35.50 for a 50-lb. bale), and there are two states of the world. In one state, the price of cotton will increase to 80 cents per pound; in the other, it will fall to 70 cents per pound. Finally, assume borrowing and lending are possible at an interest rate of 5 percent over the period. Using the tools of state-preference theory, we can set up the payoff array in table 6. There will ultimately only be two cash markets for cotton here. We can either transact now in the spot market, or we can transact later in whichever of the two subsequent spot markets is available.

The Hedger

Now consider the situation of a cotton farmer who will harvest 25 tons of cotton in the coming period. To restrict the number of contingencies, we treat the size of the crop as certain. In terms of state-claims, this is the endowment (not a transaction) described in table 7. Finally, assume the farmer is risk-averse and wants his cash receipts to be the same, regardless of the state of the world. In other words, he wants a final consumption bundle, after harvesting and selling the crop, as described in table 8. Note that we must restrict the payoff $X to be strictly greater than $35,000 here. Otherwise, the farmer could achieve a certain payoff of $35,000 by giving away $5,000 in the high-price state of the world. The potential for arbitrage implies that he should be able to do better than this. Can he achieve his desired payoff with the current set of markets?

[31]Starting from the same initial allocation, where Jack has one chop and Bubbles two, Jack buys both chops from Bubbles and then sells back to her the three fat portions.

[32]This is their proof of the existence of an efficient general equilibrium in an economy with complete markets. See Geanakoplos (1987) for an overview.

[33]See Chicago Mercantile Exchange (1983), p. 8.

Table 6

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Buy now	−$35,500	25 tons	0	0	0	0
Buy @ 70¢	0	0	25 tons	0	−$35,000	0
Buy @ 80¢	0	0	0	25 tons	0	−$40,000
Borrow cash	$100	0	0	0	−$105	−$105

Table 7

	CURRENT COST	STATE CLAIM PAYOFFS Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Endowment	0	0	25 tons	25 tons	0	0

Table 8

	CURRENT COST	STATE CLAIM PAYOFFS Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Desired Endowment	0	0	0	0	$X	$X

Table 9

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Forward sale	0	0	−25 tons	−25 tons	$X	$X

The answer is no: to convert his certain crop into a certain dollar payoff, he requires a forward sale contract of the form described in table 9. Such a contract does not exist here, nor can it be synthesized as a combination of available contracts. In terms of our earlier discussion, the system of markets is incomplete. In this case, the incompleteness prevents the farmer from arranging his desired payoff vector. Let's consider two ways of alleviating this constraint by completing the system of markets.

First, let's add storage (and leasing) of the physical commodity to the list of allowable transactions. In this example, no price is charged to store or lease cotton, although such a price could readily be included. In our simplified two-state, two-period example, this is sufficient to complete the system of markets. The payoff array is now given in table 10. To arrange the desired bundle, the farmer can now arrange the transactions in table 11, converting his endowment into a certain dollar amount next period. In other words, the farmer leases 25 tons of cotton, sells it for cash today and invests the proceeds, repaying the borrowed cotton when his own crop is harvested. Thus, it is possible, even in the absence of a futures market, to hedge

Table 10

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS: Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Buy now	−$35,500	25 tons	0	0	0	0
Buy @ 70¢	0	0	25 tons	0	−$35,000	0
Buy @ 80¢	0	0	0	25 tons	0	−$40,000
Borrow cash	$100	0	0	0	−$105	−$105
Store cotton	0	−25 tons	25 tons	25 tons	0	0

Table 11

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS: Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Lease cotton	0	25 tons	−25 tons	−25 tons	0	0
Sell now	$35,500	−25 tons	0	0	0	0
Lend	−$35,500	0	0	0	$37,275	$37,275
Total:	**0**	**0**	**−25 tons**	**−25 tons**	**$37,275**	**$37,275**

Table 12

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS: Cotton Now	Cotton Later @ 70¢	Cotton Later @ 80¢	Cash Later @ 70¢	Cash Later @ 80¢
Futures	0	0	−50 lb.	−50 lb.	P_F	P_F

the crop. This is achieved, however, through a circuitous and potentially costly chain of three transactions. Why shouldn't the farmer arrange the desired transaction directly as a single contract?

This is precisely the role of a futures contract, our second means of completing this system of markets. A futures contract would have just this payoff vector, perhaps adjusted by a scale factor. A cotton future is a standardized forward sale contract; i.e., a contract to pay a specified price (P_F) for a standardized quantity (50 lbs. of cotton) at a specific time (next period), regardless of the state of the world, as described in table 12. The futures contract makes the marketplace more flexible; in our simple example, it completes the system of markets. It allows the farmer to transfer directly the price risk associated with commodity ownership without transferring ownership of the commodity per se.

There are three lessons here. First and foremost, completing the system of markets makes everyone better off (or at least not worse off) by allowing risk to be transferred from the farmer to a speculator. Second, there is more than one way to complete an incomplete system of markets. Third, some means of completing a system of markets may be more cost-effective than others. One might even plausibly conjecture that all missing markets result from transaction costs that render them cost-ineffective. Confirming or refuting such a conjecture, however, is beyond the scope of this paper.

The Speculator

The same transaction can also be considered from the perspective of the speculator who buys a futures contract from the farmer. A speculator is someone who wagers that she can

Table 13

TRANSACTION	CURRENT COST	Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
Buy now	$-P_0C$	C	0	0	0	0
Buy @ P_L	0	0	C	0	$-P_LC$	0
Buy @ P_H	0	0	0	C	0	$-P_HC$
Store	0	−C	C	C	0	0
Invest	−$1	0	0	0	1+R	1+R

Table 14

TRANSACTION	CURRENT COST	Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
Lease	0	C	−C	−C	0	0
Sell now	P_0C	−C	0	0	0	0
Buy @ P_L	0	0	C	0	$-P_LC$	0
Buy @ P_H	0	0	0	C	0	$-P_HC$
Lend	$-P_HC/(1+R)$	0	0	0	P_HC	P_HC
Net:	**$C[P_0 - P_H/(1+R)]$**	**0**	**0**	**0**	**$C(P_H - P_L)$**	**0**

accurately predict the state of the world.[34] To see how speculation occurs in the absence of futures markets, we start with a market that allows for storage (and leasing) of cotton. If we omit cotton storage, as in table 6, then the system of markets is incomplete; the speculator is thus prevented from arranging her desired payoff vector. Once again we complete the market in two different ways: with commodity market speculation and with futures market speculation.

To make the problem more general, we now represent contract quantities (measured in lbs. of cotton) by C, and prices (measured as cents per lb. of cotton) by P. There are three prices, the current spot price P_0, low price P_L and a future high price P_H (where $P_H > P_L$). There is a single fixed interest rate R, at which the speculator can borrow and lend money. This payoff array is given in table 13. Let's say the speculator believes the low-price state will occur. In particular, she wants to arrange a transaction that pays only cash in the low price state. She can accomplish this by arranging the bundle of transactions described in table 14. This concentrates the speculator's return on the single state-claim, cash later in state P_L.

In other words, to speculate on the low-price outcome in this world, the speculator must sell the physical commodity short. This requires leasing the commodity, selling it in the spot market and repurchasing the cotton later when the price has changed. These four transactions imply a profit if the cotton price falls and a loss if it rises. Finally, the speculator invests an amount equal to the discounted value of the cotton in the high price state. If this state occurs, then the cost of repurchasing the cotton will be exactly covered by the investment; if the low price state occurs, the investment more than covers the repurchase of the cotton, and the difference is a speculative profit. The current cost of this basket is the proceeds of the spot sale, P_0C, less the amount that must be invested, $P_HC/(1+R)$. This must be negative—i.e., a positive cost to the speculator—if an arbitrage opportunity is to be avoided.

One must question the practicality of such a transaction, however. Once again, there is more than one way to complete a system of markets.

[34]Note that the speculator is not necessarily a consumer of cotton. Indeed, a cotton consumer (e. g., a clothing manufacturer) is likely to be as risk-averse as the farmer. In effect, a speculator sells insurance (i. e., bears risk) for a living.

Table 15

TRANSACTION	CURRENT COST	Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
		STATE CLAIM PAYOFFS				
Sell Standard Futures	0	0	−50 lb.	−50 lb.	P_F	P_F
Buy @ P_L	0	0	50 lb.	0	$-P_L$	0
Buy @ P_H	0	0	0	50 lb.	0	$-P_H$
Cash Settlement Futures	0	0	0	0	P_F-P_L	P_F-P_H

Although short sales are commonplace in the stock market, for example, such transactions can be considerably more difficult when dealing with physical commodities. If a market for cotton futures is available, however, the speculator can arrange her desired bundle without ever having to store or lease a physical commodity. Indeed, most futures contracts are retired by a process called *cash settlement*, which obviates any transfer of the physical commodity.

In effect, a cash settlement futures contract (table 15) is a bundle of transactions sold as a single unit, where P_F is the futures price. Note that, to eliminate arbitrage here, it must be the case that $P_F - P_L > 0 > P_F - P_H$.[35] Such a standardized contract facilitates the transfer of risk from hedgers to speculators. Moreover, with cash settlement, the speculator never has to handle the physical commodity sold in the futures contract, thus reducing transaction costs.

Options and Investor Flexibility

The preceding example illustrated how the introduction of a futures contract, a paper transaction, could improve the allocation of resources in an economy.[36] Such applications of state-preference theory are not limited to the futures markets. Arrow (1964) showed that the ability to reallocate risk without otherwise constraining economic activity is a general property of securities markets. This principle can also be seen at work in the options market.[37] In considering an options example, we abstract from the issues of timing and consumption and concentrate on uncertainty, to streamline the example.

Options markets are especially useful for shifting risks, because of a special characteristic of an option contract. In particular, a call option specifies a strike price, labeled K, at which the holder of the option can purchase the underlying commodity.[38] By simply changing the strike price in an option contract, we can create a fundamentally different financial security. Thus, a "single" options market provides the opportunity to exchange a multitude of state claims. To see how this works, consider the following example.

The Chicago Board of Trade Options Exchange (CBOE) trades options on the Standard and Poors' 500 stock index (S&P 500), among other things.[39] A call option on the S&P 500 with striking price of 295, theoretically gives the op-

[35]In general, in order to eliminate arbitrage opportunities, every payoff vector must involve a trade-off, in which some element (including the current cost) of every vector is negative and some element of every vector is positive. In other words, every vector must represent an exchange of some sort rather than a unilateral gift.

[36]In one sense, we have reallocated risk rather than resources. Recall, however, that we have redefined goods, so that cotton in the high-price state is a different resource from cotton in the low-price state.

[37]For more thorough analyses of options and complete markets, see Ross (1976), and Arditti and John (1980).

[38]The "commodity" in most options markets is really a share of a corporation's common stock. However, there are also organized options exchanges for cattle, copper, crude oil, Canadian dollars, etc. See the shaded insert on pp. 51-52 for a basic description of option contracts.

[39]The S&P 500 is a weighted average of 500 common stocks. The amount of each stock included in the index is pro-rated according to the value of that stock, where the value is defined as the price per share times the number of shares outstanding. The value of the index is then scaled down to make the base period index (1941-43) worth 10 units; if the S&P 500 today is worth 295, for example, it is worth 29.5 times as much as it was in 1943. Thus, the price of the S&P 500 is not, strictly speaking, a dollar value for the index.

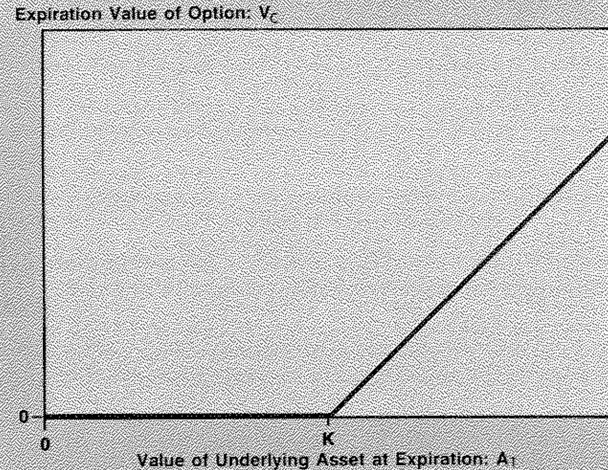
Call Option Basics

A *call option* is a legal contract that gives its owner the right to buy a specified asset at a fixed price on a specified date.[1] Similarly, a *put option* gives its owner the right to sell a specified asset at a fixed price on a specified date. Option contracts are usually sold by one party to another.[2] The person who owns an option contract is called the *holder* of the option. The person who sells an option contract —that is, the person who will be compelled to perform if the option holder invokes her right as specified in the contract—is called the *writer* of the option. The act of invoking the contract is called *exercising* the option. The fixed price identified by the option contract is called the *striking price*. The date at which the option can be exercised is called the *expiration date* of the option.

These legal contracts are probably best known by the stock options that are bought and sold by brokers in the trading pits of organized options exchanges in Chicago, New York and elsewhere. In addition to options on common stock, there are active markets for options on foreign currencies, on stock index portfolios, on government securities, and on futures for agricultural commodities, to name a few. The definition of an option, however, does not limit the term to those contracts actively traded on the floors of organized financial exchanges. By definition, an option is any appropriately constructed legal contract between the writer and the holder, regardless of whether it is actively traded.

Consider now the value to the holder of an expiring call option, as illustrated in figure 1. The value of the underlying asset specified by the contract, A_T, is given on the horizontal axis, while the value of the option itself, V_C, is given on the vertical axis. The point K on the horizontal axis is the specified striking price for the asset. If the value of the underlying asset is below the striking price on the expiration date, then the call option will not be exercised; anyone who truly wanted to buy the asset would do so outright at the going price, rather than using the option and paying the striking price. In this case, the option expires worthless, and the option holder experiences no gain or loss on the expiration date.

Figure 1
Value of Call Option to Holder



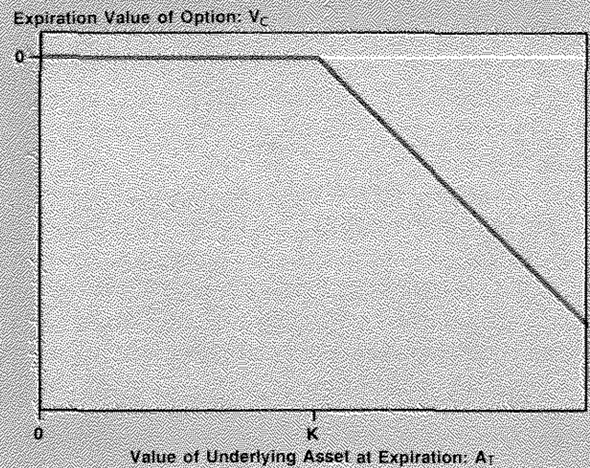
On the other hand, if the value of the asset is above the striking price, then the holder will exercise his option and pay the striking price for the asset. In this case, his net gain on the expiration date will be $(A_T - K)$, the difference between the current price and the striking price, since he can turn around and resell the asset immediately, if he wants to. Thus, the expiration value of the option and the decision about whether to exercise are contingent upon the value of the underlying asset at that time:

State	Action	Option value
$A_T < K$	No exercise	$V_c = 0$
$A_T \geq K$	Exercise	$V_c = A_T - K$

For this reason, options are also referred to as "contingent claims" on the underlying assets.

The corresponding net payoffs to the writer of the call option are given in figure 2. Notice that his payoffs are exactly the inverse of those for the option holder. Also note that the payoff at expiration to the writer of an option is never positive; at best it is zero. It is for this reason that options are sold to the holder, rather than being given away free

Figure 2
Value of Call Option to Writer



of charge. The price initially paid for the option—the option price or option premium—could be incorporated into the figures by simply shifting the holder's payoffs down and the writer's payoffs up by the appropriate amount.

[1]This definition is a paraphrase of the definition given by Cox & Rubinstein (1985), p. 1. It describes a "European" option, which is distinguished from an "American" option. An American option gives its owner the right to buy at any time on *or before* the specified date.

[2]They are sold, because options have a non-negative value; because they are a right to buy (or sell) the asset, they do not compel the owner of the contract to do anything. Although they are valuable, nothing in the definition of an option requires that they be offered for sale; that is, their value does not depend on how they were obtained.

tion holder the right to purchase the S&P 500 index at 295.[40] Because the weighting scheme involved in constructing the index involves 500 stocks, many held in fractional quantities, all of which must then be scaled down to a relative value, however, it is impossible to purchase the exact S&P 500 index. Instead, cash settlement is used: at maturity, the holder of a call option receives in cash $500 times the difference between the value of the index and the striking price, if this difference is positive, and nothing otherwise.[41]

To simplify our example, we limit the number of possible outcomes as before and ignore the time dimension. As at Portfolio Downs, there are only two relevant times: before the true outcome is revealed and afterward. Each state of the world corresponds to a different value for the S&P 500:

State	Value of S&P 500
A	286
B	294
C	299
D	301
E	306
F	314

Thus, our option to buy at 295 is a state claim. In the event that the value of the S&P 500 itself is less than 295, then the option is worthless, because the (approximate) index can be purchased in the open market for less than the strike price. If the index is worth more than 295, then the option is worth $500 times the difference between the actual price and the strike price. This is summarized in table 16.

It should be clear from our earlier examples that this system of markets is not complete. There is no way, for example, to arrange a portfolio that pays off exactly $50,000 if the S&P 500 is below 300 and zero otherwise; i.e., ($50000,$50000,$50000,0,0,0). The special characteristic of options markets, however, is that linearly independent payoff vectors can be achieved by changing the striking price alone. Because of this, numerous options on the same security are actively traded. Many of these options differ only in their striking prices. Adding some of these options to our example, we get the payoff array in table 17.

This system of markets is complete. To achieve the payoff vector ($50000,$50000,$50000,0,0,0), for example, we can transact the amounts of the six listed call options described in table 18.

[40]The units for the option contract must be rescaled to give a dollar value. In particular, option contracts are for $500 times the level of the index. For example, to exercise a call option at strike price 295 would cost $500 x 295 = $147,500.

[41]For example, an investor exercising a call option at strike price 295 when the index itself is at 290 would receive $500 x (295-290) = $2500.

Table 16

SECURITY	A	B	C	D	E	F
	PAYOFFS BY STATE					
Call option at K=295	0	0	$2000	$3000	$5500	$9500

Table 17

SECURITY	A	B	C	D	E	F
	PAYOFFS BY STATE					
Call at K @ 285	$500	$4500	$7000	$8000	$10,500	$14,500
Call at K @ 290	0	$2000	$4500	$5500	$8000	$12,000
Call at K @ 295	0	0	$2000	$3000	$5500	$9500
Call at K @ 300	0	0	0	$500	$3000	$7000
Call at K @ 305	0	0	0	0	$500	$4500
Call at K @ 310	0	0	0	0	0	$2000

Table 18

SECURITY	QUANTITY	A	B	C	D	E	F
		PAYOFFS BY STATE					
Buy calls @ 285	−100	$50,000	$450,000	$700,000	$ 800,000	$1,050,000	$1,450,000
Sell calls @ 290	200	0	−$400,000	−$900,000	−$1,100,000	−$1,600,000	−$2,400,000
Buy calls @ 295	−125	0	0	$250,000	$ 375,000	$ 687,500	$1,187,500
Sell calls @ 300	150	0	0	0	−$ 75,000	−$ 450,000	−$1,050,000
Buy calls @ 305	−625	0	0	0	0	$ 312,500	$2,812,500
Sell calls @ 310	1000	0	0	0	0	0	−$2,000,000
Totals:		$50,000	$ 50,000	$ 50,000	0	0	0

Note that the numbers of the various options bought and sold here are not dollar amounts; the total dollar cost is the price of each option times the respective quantity. Option pricing is beyond the scope of this paper. The no-arbitrage condition implies, however, that the total cost of this portfolio of options must be somewhere between zero and $50,000.

In practice, of course, there are many more than six possible prices for the S&P 500 index and far more states of the world than there are possible prices for the index portfolio. Nonetheless, the linearly independent payoffs of these six options necessarily span more of the payoff space than does the index alone. The result is greater flexibility for investors in fashioning their portfolios in an uncertain world.

CONCLUSIONS

The theory of complete markets and the parallel state-preference theory have been active areas of research in the economics and finance literature for almost 40 years. Despite their theoretical importance, these topics have received little exposure elsewhere, doubtless because of the technical nature of the argument. This paper conveys the basic concepts of the theory for a non-specialized audience.

The theory of complete markets sheds light on many economic issues. It starts by redefining goods to include attributes not normally considered inherent: the time and state at which something is consumed. An immediate implication of this new definition of a good, however, is that the system of markets available in the real world is far from complete: taking all the possible combinations of attributes into account leaves the number of goods far in excess of the number of actual markets. By the same token, this relative dearth may provide a ready explanation for the recent proliferation of unusual financial innovations.

One of the overriding themes appearing in all the examples here is the efficiency of the allocation of goods. This can be seen in the allocation of risks in futures and options markets, the ability to refine payoffs at the racetrack, or the distribution of fat and lean between Jack Sprat and his wife, Bubbles. The common implication in each example is that additional markets can improve the welfare of all concerned. That is, given the ability to reallocate, individuals will do so: they will exchange relatively less desirable commodity bundles for those that are, for them, relatively more desirable. A complete system of markets provides this ability. Thus, an economy with greater flexibility in production, consumption and investment is uniformly preferable to one with less.

A second recurring theme is uncertainty. State-preference theory and the theory of complete markets are one way to incorporate uncertainty systematically in an economic model. This is central to a theory that includes an uncertain state of the world as a fundamental attribute of a good. One result is a recognition of the value of the ability to reallocate risk through financial transactions. While speculation in financial markets may or may not be unfettered gambling, the implicit transfer of risk from hedgers to speculators still produces economic value. The theory of complete markets thus provides a systematic explanation for the popularity and value of many so-called derivative securities, such as futures and options.

Review Questions

(1) Using the payoff array in table 3, construct a book of bets that amounts to a pure security for the outcome T-C-M; i. e., make a book with the payoff vector (0,$1,0,0,0,0). What is the net investment required for this book of bets?

(2) Using the payoff array in table 3, what is the probability, implicit in the bookmaker's odds, that the outcome of the race will be T-C-M?

(3) Suppose the bookmaker offers a redundant bet with inconsistent odds. In particular, suppose the last bet in the payoff array in table 1 is changed, yielding the new array in table Q1. Construct an arbitrage portfolio (a portfolio that shows a positive net profit in *all* outcomes) from these bets.

(4) Ross shows that a necessary condition for a collection of securities to span the state space is that, "for any two states there must be some asset whose payoffs distinguish between them."[1] Suppose we have instead a market in which no asset can distinguish between two states. In particular, consider the payoff array in table Q2, in which states E and F are indistinguishable by the available assets. Show that this system of markets is incomplete.

(5) Consider a speculator in the cotton market who believes that the price will rise. Assume that storage (and leasing) of cotton is not practical, but that a cash settlement futures market exists. In other words, assume the allowable transactions are those in table Q3. What sort of payoff vector might this speculator want? How could she construct it from the available transactions?

(6) Consider a banker facing the transactions available in table Q4 over the next 30 days in the international financial markets, where S is the spot foreign exchange rate (¥/$), and $R_¥$ and $R_\$$ are the foreign and domestic 30-day interest rates. Is this system of markets complete? Suppose the banker wants to arrange a forward exchange contract of the form presented in table Q5. Arrange such a payoff from the available transactions. What does the forward rate F equal? What is the intuition behind this value for the forward rate?

[1]See Ross (1976), p. 81.

Table Q1

BET	ODDS	PAYMENT	T-M-C	T-C-M	M-T-C	M-C-T	C-T-M	C-M-T
			PAYOFFS BY OUTCOME					
T wins	7-3	– $2	$6.67	$6.67	0	0	0	0
T places	2-3	– $2	$3.33	$3.33	$3.33	0	$3.33	0
M wins	4-1	– $2	0	0	$10.00	$10.00	0	0
M places	9-11	– $2	$3.64	0	$3.64	$3.64	0	$3.64
C wins	1-1	– $2	0	0	0	0	$4.00	$4.00
C places	1-4	– $2	0	$2.50	0	$2.50	$2.50	$2.50

Table Q2

SECURITY	QUANTITY	A	B	C	D	E	F
		PAYOFFS BY STATE					
Buy calls @ 285	– 100	$50,000	$450,000	$700,000	$ 800,000	$1,450,000	$1,450,000
Sell calls @ 290	200	0	– $400,000	– $900,000	– $1,100,000	– $2,400,000	– $2,400,000
Buy calls @ 295	– 125	0	0	$250,000	$ 375,000	$1,187,500	$1,187,500
Sell calls @ 300	150	0	0	0	– $ 75,000	– $1,050,000	– $1,050,000
Buy calls @ 305	– 625	0	0	0	0	$1,250,000	$1,250,000
Sell calls @ 310	1000	0	0	0	0	– $2,000,000	– $2,000,000

Table Q3

TRANSACTION	CURRENT COST	Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
		STATE CLAIM PAYOFFS				
Sell Futures	0	0	0	0	$P_FC - P_LC$	$P_FC - P_HC$
Sell now	P_0C	– C	0	0	0	0
Buy @ P_L	0	0	C	0	$-P_LC$	0
Buy @ P_H	0	0	0	C	0	$-P_HC$
Lend	– $1	0	0	0	1 + R	1 + R

Table Q4

CONTRACT	$ NOW	¥ now	¥ 30-day	$ 30-day
		PAYOFFS TO CONTRACTS		
Buy spot ¥	– $1	S	0	0
Convert & lend ¥	– $1	0	$S(1 + R_¥)$	0
Lend $	– $1	0	0	$1 + R_\$$

Table Q5

CONTRACT	$ NOW	¥ now	¥ 30-day	$ 30-day
		PAYOFFS TO CONTRACTS		
Buy forward ¥	$0	0	F	– $1

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Answers to the Review Questions

(1) Try:(30¢, − 60¢,20¢, − 25¢,100¢, − 85¢). The net investment is 20¢.

(2) Odds are given by:

$$O_e = \frac{1.00 - .20}{.20} = 4 = \text{4-to-1 odds.}$$

$$O_e = \frac{1}{P(e)} - 1 = 4 \qquad P(e) = .20 = 20\%.$$

(3) Try: ($240, − $240,$160, − $220,$400, − $320).

(4) Compare: (0,0,0,0,1,0) with: (0,0,0,0,0,1), where the elements of the vectors represent the numbers of each option to buy or sell. The last two options are redundant: there are six states and five non-redundant securities—the system of markets is incomplete.

(5) She would want to purchase a payoff $Z > 1$ in the high-price state, as described in table A1. This could be obtained by the transactions listed in table A2.

(6) Yes, it is complete. Combine the transactions listed in table A3. Thus, $F = S(1 + R_¥)/(1 + R_\$)$. This is simply a statement of the covered interest parity condition, $F(1 + R_\$) = S(1 + R_¥)$, which prevents arbitrage between money markets in different countries.

Table A1

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS: Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
Desired Payoffs	−$1	0	0	0	0	$Z

Table A2

TRANSACTION	CURRENT COST	STATE CLAIM PAYOFFS: Cotton Now	Cotton Later @ P_L	Cotton Later @ P_H	Cash Later @ P_L	Cash Later @ P_H
Buy futures:	0	0	0	0	P_LC-P_FC	P_HC-P_FC
Lend:	$-C(P_F-P_L)/(1+R)$	0	0	0	P_FC-P_LC	P_FC-P_LC
Net:	$-C(P_F-P_L)/(1+R)$	**0**	**0**	**0**	**0**	$C(P_H-P_L)$
Scaling this down, it becomes:						
Net:	**−$1**	**0**	**0**	**0**	**0**	$(1+R)\left[\frac{(P_H-P_L)}{(P_F-P_L)}\right]$

Table A3

CONTRACT	$ NOW	PAYOFFS TO CONTRACTS: ¥ now	¥ 30-day	$ 30-day
Convert & lend ¥	$\frac{-\$1}{1+R_\$}$	0	$S\frac{1+R_¥}{1+R_\$}$	0
Borrow $	$\frac{\$1}{1+R_\$}$	0	0	−$1
Net:	**0**	**0**	**F**	**−$1**