# THE STRUCTURE OF $m$–STABLE SETS AND IN PARTICULAR OF THE SET OF RISK NEUTRAL MEASURES (PRELIMINARY VERSION, WILL BE UPDATED LATER)

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Abstract. The study of dynamic coherent risk measures and risk adjusted values as introduced by Artzner, Delbaen, Eber, Heath and Ku, leads to a property called fork convexity. We give necessary and sufficient conditions for a closed convex set of measures to be fork convex. Since the set of martingale measures for price processes is fork convex, this leads to a characterisation of closed convex sets that can be obtained as the set of risk neutral measures in an arbitrage free model of security prices. We also relate the property of fork convexity or m–stability with the validity of Bellman's principle. It turns out that the stability property investigated in this paper is equivalent to properties known as time-consistency and rectangularity as used in multiprior Bayesian decision theory.

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*Key words and phrases.* arbitrage theory, capital requirement, coherent risk measure, capacity theory, dynamic risk measures, martingale measures, multiple priors, rectangularity, risk neutral measures, Snell envelope, shortfall, submartingale method, time consistency.

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## 1. Introduction and Notation

The concept of coherent risk measures together with its axiomatic characterization was introduced in the paper [ADEH1] and further developed in [ADEH2] and [Delb]. The idea of dynamic coherent risk measures or parallel to it, dynamic risk adjusted values was introduced in [ADEHK]. A characterisation of the risk measures defined on the space of càdlàg processes is given by Cheridito-Delbaen-Kupper [CDK1], [CDK2] and [CDK3] . The relation between their theory and the present paper is the subject of ongoing reserach, [Kupper]. Some of the examples given in [ADEHK] require the use of sets satisfying a property that is called multiplicative stability. Another name for the same concept is fork convexity, a terminology that was introduced by Zitkovic [Zit]. In decision theory this property is known as rectangularity, see Epstein and Schneider, Wang ([ES],[Wang]). These papers deal with the case of finite $\Omega$. In turns out that there are natural examples of multiplicatively stable convex sets. One of these examples is the set of absolutely continuous risk neutral measures for an arbitrage free price process, see below for precise statements. In this paper we give necessary and sufficient conditions for a closed convex set of measures to satisfy this stability property. The conditions are related to concepts such as "price of risk" and fit well in economic theory. Applying this characterisation to the situation of arbitrage free price processes, we will give a characterisation of those sets that can arise as sets of risk neutral measures. Especially in the case of filtrations where all martingales are continuous, we will solve the problem completely. The more general case will be the subject of a later paper.

Throughout the paper, we will work with a fixed, filtered probability space, denoted as $\left(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. The filtration $\mathcal{F}$ is supposed to satisfy the usual assumptions, i.e. the filtration is right continuous and $\mathcal{F}_0$ contains all the null sets of the complete sigma–algebra $\mathcal{F}_\infty$. The time set is supposed to be $\mathbb{R}_+$. The reader can check that this is the most general case. By using suitable imbeddings it covers the case of discrete, finite as well as infinite, time sets. With $\mathbf{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ (or $\mathbf{L}^\infty(\mathbb{P})$ or even $\mathbf{L}^\infty$ if no confusion is possible), we mean the space of all equivalence classes of bounded real valued random variables. The space $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ (or $\mathbf{L}^0(\mathbb{P})$ or simply $\mathbf{L}^0$) denotes the space of all equivalence classes of real valued random variables. The space $\mathbf{L}^0$ is equipped with the topology of convergence in probability. The space $\mathbf{L}^\infty(\mathbb{P})$, equipped with the usual $\mathbf{L}^\infty$ norm, is the dual space of the space of integrable (equivalence classes of) random variables, $\mathbf{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ (also denoted by $\mathbf{L}^1(\mathbb{P})$ or $\mathbf{L}^1$ if no confusion is possible). The spaces $\mathbf{L}^p$ for $0 < p < \infty$ are defined in the usual way. A useful result in integration theory is the so-called Scheffé's lemma. It says that if a sequence of nonnegative random variables $f_n$ tends in probability to a random variable $f$, if moreover $\mathbb{E}[f_n]$ tends to $\mathbb{E}[f] < \infty$, then necessarily the convergence takes place in $\mathbf{L}^1$ and the sequence is therefore uniformly integrable. We will frequently use this lemma.

In the general theory of stochastic processes, stochastic intervals play a special role. If $T \leq S$ are two stopping times, then the stochastic intervals are defined as follows

$$[\![T, S]\!] = \{(t,\omega) \mid t \in \mathbb{R}_+ \text{ and } T(\omega) \leq t \leq S(\omega)\}\,.$$

The other intervals are defined in a similar way. In case $T = S$ we simply write $[\![T, S]\!] = [\![T]\!] = \{(t,\omega) \mid T(\omega) < \infty\}$. If $T$ is a stopping time and if $A \in \mathcal{F}_T$, then

$T_A$ denotes the stopping time defined as $T_A = T$ on the set $A$ and $T_A = \infty$ on the set $A^c = \Omega \setminus A$. In particular for $t \in \mathbb{R}_+$ and $A \in \mathcal{F}_t$ we have $[\![t_A]\!] = \{t\} \times A$.

With the given filtration we will construct the sigma–algebras of predictable and optional sets. The predictable sigma–algebra, denoted by $\mathcal{P}$, is the smallest sigma–algebra on $\mathbb{R}_+ \times \Omega$ that contains sets of the form $[\![0_A]\!] = \{0\} \times A$ with $A \in \mathcal{F}_0$, as well as for each stopping time $T$, the stochastic interval

$$[\![0, T]\!] = \{(t, \omega) \mid t \leq T(\omega) \text{ and } t < \infty\}.$$

The optional sigma–algebra, denoted by $\mathcal{O}$, is the smallest sigma–algebra on $\mathbb{R}_+ \times \Omega$ that contains sets of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$, as well as for each stopping time $T$, the stochastic interval

$$[\![0, T[\![= \{(t, \omega) \mid t < T(\omega)\}.$$

We remark that the indicator functions of elements of the generating set of $\mathcal{P}$ are left continuous adapted processes and that the indicator functions of elements of the generating sets of $\mathcal{O}$ are right continuous adapted processes. For these notions from the general theory of stochastic processes, we refer the reader to [DM]. It can easily be checked that $\mathcal{P} \subset \mathcal{O}$.

Let us recall that the class of predictable sets

$$\{[\![0_A]\!] \mid A \in \mathcal{F}_0\} \cup \{]\!]T, S]\!] \mid T \leq S \text{ stopping times}\},$$

forms a semi–algebra that generates $\mathcal{P}$. The Boolean algebra generated by this class is simply

$$\mathcal{A} = \Big\{[\![0_A]\!] \cup ]\!]T_0, T_1]\!] \cup ]\!]T_1, T_2]\!] \ldots \cup ]\!]T_{n-1}, T_n]\!] \\ \mid n \geq 1; A \in \mathcal{F}_0 \text{ and } 0 \leq T_0 \leq T_1 \leq \ldots T_n \leq +\infty \text{ are all stopping times}\Big\}.$$

The importance of this class lies in the following density result from general measure theory. The proof of the lemma is included in the proof of the Carathéodory extension theorem.

**Lemma 1.1.** *Let $\mu$ be a nonnegative finite sigma–additive measure on $\mathcal{P}$, then for each $\varepsilon > 0$ and for each set $B \in \mathcal{P}$, there is a set $A \in \mathcal{A}$ such that $\mu(A \Delta B) \leq \varepsilon$.*

If $\mathbb{Q}$ is a probability defined on the $\sigma$-algebra $\mathcal{F}_\infty$, we will use the notation $\mathbb{E}_\mathbb{Q}$ or $\mathbb{Q}$, to denote the expected value operator defined by the probability $\mathbb{Q}$. So we will write $\mathbb{E}_\mathbb{Q}[f]$ or $\mathbb{Q}[f]$ to denote the expected value of $f$. Since the filtration satisfies the usual assumptions, we will suppose that all the (sub–, super–) martingales are càdlàg, meaning they are right continuous and have left limits. When we deal with the construction of the Snell envelope, we will pay attention to this continuity property and the reader will notice similar difficulties as in the work of Mertens see [M] and [DM], appendix. We will identify, through the Radon–Nikodym theorem, finite measures $\nu$ on $\mathcal{F}_\infty$, that are absolutely continuous with respect to $\mathbb{P}$, with their densities $\frac{d\nu}{d\mathbb{P}}$, i.e. with functions in $\mathbf{L}^1$. Furthermore we will sometimes identify this measure with the càdlàg martingale $Z_t = \mathbb{E}_\mathbb{P}\left[\frac{d\nu}{d\mathbb{P}} \mid \mathcal{F}_t\right]$. We hope that these identifications will not cause too many problems.

We can now state the definition of multiplicatively stable sets. The definition is related to the concept of stable sets as in [DM]. To simplify the writing of the definition we suppose that $\mathcal{S}$ is a set of probability measures, all elements of which are absolutely continuous with respect to $\mathbb{P}$. The elements $\mathbb{Q}$ of the set $\mathcal{S}$ will (as already said above) be identified with their Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and therefore we see $\mathcal{S}$ as a subset of $\mathbf{L}^1$ Most of the time, the set $\mathcal{S}$ will be supposed to be convex, also we will always have that $\mathbb{P} \in \mathcal{S}$. In that case we have for all $\varepsilon > 0$ and all $\mathbb{Q} \in \mathcal{S}$ that $(1-\varepsilon)\mathbb{Q} + \varepsilon\mathbb{P} \in \mathcal{S}$. This means that every element in $\mathcal{S}$ can be approximated (in $\mathbf{L}^1$-norm) by elements in $\mathcal{S}$ that are also equivalent to $\mathbb{P}$. The set of elements in $\mathcal{S}$ that are also equivalent to $\mathbb{P}$ is denoted by $\mathcal{S}^e$. If $\mathbb{Q} \in \mathcal{S}^e$ then the martingale $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$ has the property $\inf_{t\in\mathbb{R}_+} Z_t > 0$, $\mathbb{P}$ a.s. (see [DM] page 85). If $\mathbb{Q} \sim \mathbb{P}$, Bayes' rule implies that $\mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_T] = \mathbb{E}_\mathbb{P}[f\, \frac{Z_\infty}{Z_T} \mid \mathcal{F}_T]$.

*Standing assumption and notation.* We will always assume that $\mathbb{P} \in \mathcal{S}$ and if $Z$ is a nonnegative (local) martingale, the expression that $Z$ is positive (we will rather say strictly positive to avoid linguistic difficulties) means that $Z_\infty > 0$ a.s. . As a consequence we have that if $Z$ is stricly positive then we have that a.s. : $\inf_t Z_t > 0$. The latter is of course stronger than $Z_t > 0$ a.s. for every $t \geq 0$.

**Definition 1.2.** *We say that a set of probability measures $\mathcal{S} \subset \mathbf{L}^1$, is multiplicatively stable, (m–stable for short) if for elements $\mathbb{Q}^0 \in \mathcal{S}, \mathbb{Q} \in \mathcal{S}^e$ with associate martingales $Z^0_t = \mathbb{E}\left[\frac{d\mathbb{Q}^0}{d\mathbb{P}} \mid \mathcal{F}_t\right]$ and $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$, and for each stopping time $T$, the element $L$ defined as $L_t = Z^0_t$ for $t \leq T$ and $L_t = Z^0_T Z_t/Z_T$ for $t \geq T$ is a martingale that defines an element in $\mathcal{S}$. We also assume that every $\mathcal{F}_0-$measurable nonnegative function $Z_0$ such that $\mathbb{E}_\mathbb{P}[Z_0] = 1$, defines an element $d\mathbb{Q} = Z_0\, d\mathbb{P}$ that is in $\mathcal{S}$.*

*Remark.* The reader can check that indeed $\mathbb{E}[L_\infty] = 1$.

*Remark.* The second part of the definition is required to be sure that when $\mathcal{F}_0$ is not trivial the set $\mathcal{S}$ is big enough. That part of the definition does not follow from the concatenation property. In most of the cases the sigma–algebra $\mathcal{F}_0$ will be trivial and then the assumption only implies that $\mathbb{P} \in \mathcal{S}$.

*Remark.* If the set $\mathcal{S}$ is m–stable and closed in $\mathbf{L}^1$, it also satisfies the property: for elements $\mathbb{Q}^0 \in \mathcal{S}, \mathbb{Q} \in \mathcal{S}^e$ with associated martingales $Z^0_t = \mathbb{E}\left[\frac{d\mathbb{Q}^0}{d\mathbb{P}} \mid \mathcal{F}_t\right]$ and $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$, and for each *predictable* stopping time $T$, the element $L$ defined as $L_t = Z^0_t$ for $t < T$ and $L_t = Z^0_{T-} Z_t/Z_{T-}$ for $t \geq T$, is a martingale that defines an element in $\mathcal{S}$. On the set $\{T = 0\}$, $Z_{T-}$ is, as usual, taken to be equal to $Z_0$. The proof that $L \in \mathcal{S}$ is quite straightforward. If $T_n$ is a sequence that announces $T$ then $L^n_t = Z^0_t$ for $t \leq T_n$ and $L^n_t = Z^0_{T_n} Z_t/Z_{T_n}$ for $t \geq T_n$ defines elements in $\mathcal{S}$. Because $T$ is predictable, we have that on the set $\{T > 0\}$, $Z_{T_n} = \mathbb{E}_\mathbb{P}[Z_\infty \mid \mathcal{F}_{T_n}]$ tends to $\mathbb{E}_\mathbb{P}[Z_\infty \mid \mathcal{F}_{T-}] = Z_{T-}$, whereas on the set $\{T = 0\}$, we always have that $Z_{T_n} = Z_0 = Z_{T-}$. It is now clear that $L^n_\infty$ tends to $L_\infty$ a.s. and therefore also in $\mathbf{L}^1$ by Scheffé's lemma. Since $\mathcal{S}$ is closed this implies $L \in \mathcal{S}$.

*Remark.* The reader familiar with the concept of stable sets of martingales, can see the resemblance between the concept of being m–stable and the usual concept of stable spaces. In a later section the reader will find why sometimes this concept is also called fork convexity.

*Remark.* Let us now analyse how to concatenate two elements in $\mathbb{Q}^0, \mathbb{Q} \in \mathcal{S}$ that are only absolutely continuous (and not necessarily equivalent) to $\mathbb{P}$. We suppose that the set $\mathcal{S}$ is convex. For each $1 > \varepsilon > 0$, let us define the probability $\mathbb{Q}^\varepsilon = \varepsilon\mathbb{P} + (1-\varepsilon)\mathbb{Q} \in \mathcal{S}$. The associated martingales are denoted by $Z^0, Z$ and $Z^\varepsilon$. If $T$ is a stopping time we define $L_t^\varepsilon$ as above, namely for $t < T$ we put $L_t^\varepsilon = Z_t^0$ and for $t \geq T$ we put $L_t^\varepsilon = Z_T^0 Z_t^\varepsilon / Z_T^\varepsilon$. On the set $\{Z_T > 0\}$ we have that $L_\infty^\varepsilon$ tends to $L_\infty = Z_T^0 Z_\infty / Z_T$ and on the set $\{Z_T = 0\}$, we must have that $Z_t = 0$ for all $t \geq T$ and hence we have that $L_\infty^\varepsilon$ tends to $L_\infty = Z_T^0$. We still have that $\mathbb{E}_\mathbb{P}[L_\infty] = 1$. Indeed

$$\begin{aligned}
\mathbb{E}_\mathbb{P}[L_\infty] &= \mathbb{E}_\mathbb{P}[Z_T^0 Z_\infty / Z_T \mathbf{1}_{\{Z_T>0\}}] + \mathbb{E}_\mathbb{P}[Z_T^0 \mathbf{1}_{\{Z_T=0\}}] \\
&= \mathbb{E}_\mathbb{P}\left[Z_T^0 \mathbf{1}_{\{Z_T>0\}} \mathbb{E}_\mathbb{P}[Z_\infty / Z_T \mid \mathcal{F}_T]\right] + \mathbb{E}_\mathbb{P}[Z_T^0 \mathbf{1}_{\{Z_T=0\}}] \\
&= \mathbb{E}_\mathbb{P}[Z_T^0 \mathbf{1}_{\{Z_T>0\}}] + \mathbb{E}_\mathbb{P}[Z_T^0 \mathbf{1}_{\{Z_T=0\}}] \\
&= \mathbb{E}_\mathbb{P}[Z_T^0] = 1.
\end{aligned}$$

It seems that the calculations are done as in the case where $\mathbb{Q} \in \mathcal{S}^e$ but with the extra notation that on the set $\{Z_T = 0\}$ we put, in a naive way, $Z_\infty / Z_T = 1$.

We will frequently use stochastic exponentials. For strictly positive martingales $Z$, with $Z_0 = 1$ — such as density processes of measures $\mathbb{Q}$ that are equivalent to $\mathbb{P}$ — we can take the stochastic logarithm defined as $N = \frac{1}{Z_-} \cdot Z$. This stochastic integral is always defined and we have that $Z = \mathcal{E}(N)$ where $\mathcal{E}$ is the stochastic exponential or Doléans-Dade exponential (see [Pr] for precise definitions).

The main theorem of this paper deals with the structure of m–stable convex closed sets $\mathcal{S} \subset \mathbf{L}^1$ of probability measures. Before we state the theorem, let us give an example of such a set (the proof that such sets are indeed m–stable is deferred). First let us recall what is usually called a multivalued mapping. For each $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we give a nonempty closed convex set $C(t, \omega)$ of $\mathbb{R}^d$. The graph of $C$ is then the set $\{(t, \omega, x) \mid x \in C(t, \omega)\}$. Set-theoretically we can identify the graph of $C$ with $C$ itself. In case the sets $C(t, \omega)$ are one–point sets, the object $C$ simply defines a mapping from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^d$. In our, more general, case we say that $C$ is a *multivalued mapping* from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^d$. We realise that from the set–theoretic viewpoint this terminology is horrible. However it is quite standard and it is widely used in the literature. We will make use of the integration theory for multivalued mappings later on. The multivalued mapping $C$ is called predictable if the graph of $C$ belongs to the product sigma–algebra $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel sigma–algebra of $\mathbb{R}^d$.

The following result gives a method to construct m–stable sets. The statement uses the technical assumption of what we call the predictable range of a sigma–martingale. This concept is explained in the appendix. The concept is needed to deal with predictable processes $q$ that are not identically zero but are such that the stochastic integral $q \cdot M$ is zero.

**Theorem 1.3.** *Let $C$ be a predictable convex closed multivalued mapping from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^d$. Let an $\mathbb{R}^d$–valued martingale $M$ be given. Suppose that for each*

$(t,\omega)$, $0 \in C(t,\omega)$ *and suppose that the projection of* $C$ *on the predictable range of the process* $M$ *is closed. Then the* $\mathbf{L}^1$*–closure* $\mathcal{S}$ *of the set*

$$\mathcal{S}^e = \left\{ \mathcal{E}(q\cdot M)_\infty \;\middle|\; \begin{array}{l} q \textit{ is predictable} \\ q(t,\omega) \in C(t,\omega) \\ \mathcal{E}(q\cdot M) \textit{ a strictly positive uniformly integrable martingale} \end{array} \right\}$$

*is an m–stable convex closed set such that* $\mathbb{P} \in \mathcal{S}$. *Furthermore the set* $\{\mathbb{Q} \in \mathcal{S} \mid \mathbb{Q} \sim \mathbb{P}\}$ *is precisely the set* $\mathcal{S}^e$ *defined above.*

*Remark.* There are two extreme cases that deserve attention. The first case is when $C(t,\omega) = \{0\}$ in which case we have that $\mathcal{S} = \{\mathbb{P}\}$. The second case is when $C(t,\omega) = \mathbb{R}^d$ in which case we have that $\mathcal{S}$ is the set of all absolutely continuous probability measures $\mathbb{Q}$, whose density process $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$, is a stochastic integral with respect to the martingale $M$.
In case the m–stable set $\mathcal{S}$ has only elements of the form $\mathcal{E}(q\cdot M)$, where $M$ is a *continuous* martingale, we can also prove an converse to the preceding result.

**Theorem 1.4.** *Let* $\mathcal{S} \subset \mathbf{L}^1$ *be an m–stable convex closed set of probability measures such that* $\mathbb{P} \in \mathcal{S}$. *Suppose that there is an* $\mathbb{R}^d$*−valued continuous martingale* $M$ *such that for each* $\mathbb{Q} \in \mathcal{S}^e$ *there is a predictable,* $\mathbb{R}^d$*−valued process* $q$ *such that* $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right] = \mathcal{E}(q\cdot M)_t$. *Then there is a predictable, convex closed multivalued mapping* $C$ *from* $\mathbb{R}_+ \times \Omega$ *into* $\mathbb{R}^d$ *such that* $0 \in C(t,\omega)$ *and such that*

$$\mathcal{S}^e = \left\{ \mathcal{E}(q\cdot M)_\infty \;\middle|\; \begin{array}{l} q \textit{ is predictable} \\ q(t,\omega) \in C(t,\omega) \\ \mathcal{E}(q\cdot M) \textit{ a strictly positive uniformly integrable martingale} \end{array} \right\}$$

Of course the latter theorem is more difficult since we have to find the multivalued mapping $C$. This will be done through the theory of multivalued measures and their corresponding Radon-Nikodym theorems. This theory was developed during the end of the sixties and is fundamental in control theory and in mathematical economics. Let us briefly describe what we will need from this theory.

Let us suppose that $(G,\mathcal{G},\mu)$ is a probability space. For a multivalued measurable mapping, $C$ from $G$ into $\mathbb{R}^d$, we define the integral of $C$ as follows

$$\int_G C(g)\mu(dg) = \left\{ \int_G q(g)\mu(dg) \;\middle|\; \begin{array}{l} q(g) \in C(g), \mu \text{ a.e.} \\ q \text{ is integrable} \end{array} \right\}.$$

It turns out that if $\mu$ is atomless, then, by the Lyapunov theorem, the integral is automatically a convex set. The existence of elements $q$ that are measurable follows from the measurable selections theorems, see [Aum] for the integration theory of set–valued mappings. The completeness of the probability space is not really needed. However in case the measure space is not complete, there are not necessarily measurable selections. The best one can obtain is an almost everywhere selection that is measurable. The existence of *integrable* selections has to be dealt with through boundedness conditions on the sets $C(g)$. A mapping $\Phi$ that assigns with each element $A$ from $\mathcal{G}$, a set $\Phi(A) \subset \mathbb{R}^d$, is called a set–valued measure if whenever $A = \cup_n A_n$ is a union of pairwise disjoint sets in $\mathcal{G}$, we can write that

$$\begin{aligned}\Phi(A) &= \sum_n \Phi(A_n) \\ &= \left\{\sum x_n \mid x_n \in \Phi(A_n) \text{ the sum being absolutely convergent}\right\}.\end{aligned}$$

The set-valued measure $\Phi$ is called $\mu$ absolutely continuous if $\mu(A) = 0$ implies that $\Phi(A) = \{0\}$. We say that $C$ is the Radon-Nikodym derivative of $\Phi$ if for each $A$ we have $\int_A C(g)\mu(dg) = \Phi(A)$. In case the set–valued measure is bounded, convex and closed valued, the absolute continuity of $\Phi$ with respect to $\mu$ guarantees the existence of a Radon-Nikodym derivative. This is a consequence of the theorem of Debreu–Schmeidler ([DeS]). In case the set–valued measure is not convex compact valued, the situation is different. The reader can consult Debreu–Schmeidler ([DeS]) and Artstein's paper, ([Art]) to have an idea of the difficulties that arise.

## 2. Elementary Stability Properties of m-Stable sets.

The definition of m–stable sets allows for immediate extensions. More precisily we have the following property, that may explain why m–stable sets are also closed fork convex.

**Proposition 2.1.** *Let $\mathcal{S} \subset \mathbf{L}^1$ be an m–stable set. Let $Z^0, Z^1, \dots Z^n$ be density processes that are elements of $\mathcal{S}^e$. Suppose that $T$ is a stopping time and suppose that $A_1, \dots, A_n$ are elements of $\mathcal{F}_T$ that form a partition of $\Omega$. The element*

$$\begin{aligned}L_t &= Z^0_t \quad \text{if} \quad t \le T \\ &= \sum_{k=1}^n \mathbf{1}_{A_k} Z^0_T \frac{Z^k_t}{Z^k_T} \quad \text{if} \quad t \ge T,\end{aligned}$$

*defines an element of $\mathcal{S}$.*

*Proof.* The proof is by induction on $n$. Let us put $L^0 = Z^0$ and for $k \ge 1$ let us define $L^k_t = Z^0_t$ if $t \le T$ and if $t \ge T$ let us put

$$\begin{aligned}L^k_t &= \mathbf{1}_{A^c_k} L^{k-1}_t + \mathbf{1}_{A_k} L^{k-1}_T \frac{Z^k_t}{Z^k_T} \\ &= \sum_{j=1}^k \mathbf{1}_{A_j} Z^0_T \frac{Z^k_t}{Z^k_T} + \mathbf{1}_{\cup_{j>k} A_j} Z^0_t.\end{aligned}$$

From the definition of m–stable sets, applied to the stopping time

$$T_{A_k} = T\,\mathbf{1}_{A_k} + \infty\,\mathbf{1}_{A^c_k},$$

it follows that if $L^{k-1} \in \mathcal{S}$, then also $L^k \in \mathcal{S}$. An induction argument now shows that $L = L^n \in \mathcal{S}$. $\square$

**Corollary 2.2.** *Let $\mathcal{S} \subset \mathbf{L}^1$ be an ($\mathbf{L}^1-$)closed m–stable set. Let $Z^0 \in \mathcal{S}$ and let $Z^n, n \geq 1$ be density processes that are elements of $\mathcal{S}^e$. Suppose that $T$ is a stopping time and suppose that $A_n, n \geq 1$ are elements of $\mathcal{F}_T$ that form a partition of $\Omega$. The element*

$$\begin{aligned} L_t &= Z_t^0 \quad \text{if} \quad t \leq T \\ &= \sum_{k \geq 1} \mathbf{1}_{A_k} Z_T^0 \frac{Z_t^k}{Z_T^k} \quad \text{if} \quad t \geq T, \end{aligned}$$

*defines an element of $\mathcal{S}$.*

*Proof.* This is easily seen. We define exactly as in the proof of the proposition, the sequence $L^k$. It is clear that $\mathbb{E}[L^k_\infty] = 1$ and by conditioning on the $\sigma-$algebra $\mathcal{F}_T$ it also follows that $\mathbb{E}[L_\infty] = \mathbb{E}[\,\mathbb{E}[L_\infty \mid \mathcal{F}_T]\,] = \mathbb{E}[L^0_T] = 1$. Furthermore we have that $L^k_\infty \to L_\infty$ a.s. . From this and Scheffé's lemma it follows that $L^k_\infty \to L_\infty$ in the $\mathbf{L}^1-$norm, implying that $L \in \mathcal{S}$. □

We will now prove the

**Theorem 2.3.** *Let $C$ be a predictable convex closed multivalued mapping from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^d$. Let an $\mathbb{R}^d-$valued martingale $M$ be given. Suppose that for each $(t, \omega)$, $0 \in C(t, \omega)$ and suppose that the projection of $C$ on the predictable range of the process $M$ is closed. Then the $\mathbf{L}^1$ closure $\mathcal{S}$ of the set*

$$\mathcal{S}^e = \left\{ \mathcal{E}(q \cdot M)_\infty \;\middle|\; \begin{array}{l} q \text{ is predictable} \\ q(t,\omega) \in C(t,\omega) \\ \mathcal{E}(q \cdot M) \text{ a strictly positive uniformly integrable martingale} \end{array} \right\}$$

*is an m–stable convex closed set such that $\mathbb{P} \in \mathcal{S}$. Moreover $\{\mathbb{Q} \in \mathcal{S} \mid \mathbb{Q} \sim \mathbb{P}\} = \mathcal{S}^e$ (as the notation suggests).*

*Proof.* The proof mainly follows from Itô's lemma. First of all it is trivial that $\mathbb{P} \in \mathcal{S}$. Let $Z^1 = \mathcal{E}(q^1 \cdot M)$ and $Z^2 = \mathcal{E}(q^2 \cdot M)$ be two strictly positive uniformly integrable martingales coming from elements in $\mathcal{S}^e$. Let $0 < \alpha < 1$ be fixed. Put $Z = \alpha Z^1 + (1 - \alpha) Z^2$, which is a strictly positive uniformly integrable martingale. We have to show that $Z_\infty \in \mathcal{S}^e$. Itô's lemma gives

$$dZ_t = \alpha Z^1_{t-} q^1_t \, dM_t + (1 - \alpha) Z^2_{t-} q^2 \, dM_t$$

We can proceed as follows

$$\begin{aligned} dZ_t &= Z_{t-} \left( \frac{\alpha Z^1_{t-}}{Z_{t-}} q^1_t + \frac{(1-\alpha) Z^2_{t-}}{Z_{t-}} q^2_t \right) dM_t \\ &= Z_{t-} q_t \, dM_t, \end{aligned}$$

where

$$q_t = \left( \frac{\alpha Z^1_{t-}}{Z_{t-}} q^1_t + \frac{(1-\alpha) Z^2_{t-}}{Z_{t-}} q^2_t \right)$$

is in the set $C$ since it is a convex combination of two elements in $C$. This proves convexity. Since $0 \in C$ we have that $\mathbb{P} \in \mathcal{S}^e$. We still have to show that elements of the closure of $\mathcal{S}^e$ and that are equivalent to $\mathbb{P}$ are of the form stated in the description of $\mathcal{S}^e$. To do this, consider a sequence $Z^n \in \mathcal{S}$ and suppose that $Z^n$ converges to the strictly positive martingale $Z$. Each $Z^n$ can be written as

$$Z^n = \mathcal{E}\left(q^n \cdot M\right),$$

where $q^n \in C$. The sequence $q^n$ does not have to converge but its projection onto the predictable range of $M$ do. Indeed we have that a.s. the brackets

$$\left[(q^n - q^m) \cdot M, (q^n - q^m) \cdot M\right]_\infty$$

tend to zero in the space of nonnegative definite matrices. It follows that there is a vector valued predictable process $q'$ such that $q^n \cdot M$ tends to $q' \cdot M$ in the space of local martingales. But the hypothesis that the projection of $C$ onto the predictable range of $M$ is closed, implies (together with the measurable selection theorem) that $q'$ is the projection of a predictable selection $q$ of the set valued mapping $C$. This implies that $Z = \mathcal{E}(q \cdot M)$ as desired. □

## 3. The Characterisation of $m-$Stable Sets in the Continuous Case.

In this section we suppose that the set $\mathcal{S}$ is a closed convex $m-$stable set. Furthermore we suppose that there is a *continuous* $\mathbb{R}^d-$valued martingale $M$ so that each element $\mathbb{Q}$ of $\mathcal{S}^e$ can be written as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(q \cdot M),$$

where $q$ is an $\mathbb{R}^d-$valued predictable process. Such a situation occurs when there is a finite dimensional martingale that has the predictable representation property. But for the moment we do not need this more restrictive assumption. The main object of this section is to prove theorem 1.4 of the introduction. As the reader can verify, it does not harm to suppose that the bracket of the martingale $M$ is bounded by 1, i.e. $Trace\langle M, M\rangle_\infty < 1$. If this is not the case we may replace $M$ by the martingale defined by the stochastic integral

$$\int \frac{1}{1 + \exp\left(200\; Trace\langle M, M\rangle\right)}\, dM.$$

This assumption simplifies the notation of the proof considerably. Before we prove the theorem let us make the precise statement (including the simplifications we introduced).

**Theorem 3.1.** *Let $\mathcal{S} \subset \mathbf{L}^1$ be an $m$–stable convex closed set of probability measures. Suppose that there is an $\mathbb{R}^d-$valued continuous martingale $M$, verifying $M_0 = 0$ and such that for each $\mathbb{Q} \in \mathcal{S}^e$ there is a predictable, $\mathbb{R}^d-$valued process $q$ such that $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right] = \mathcal{E}(q \cdot M)_t$. Suppose further that $Trace\langle M, M\rangle_\infty < 1$. Then there is a predictable, convex closed set-valued mapping $C$ from $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}^d$ such that $0 \in C(t, \omega)$ and such that*

$$\mathcal{S}^e = \left\{ \mathcal{E}(q \cdot M)_\infty \,\middle|\, \begin{array}{l} q \textit{ is predictable} \\ q(t,\omega) \in C(t,\omega) \\ \mathcal{E}(q \cdot M) \textit{ is a strictly positive integrable martingale} \end{array} \right\}.$$

The proof is divided into several steps. Because of this we will introduce an extra notation. The finite measure $\mu$ on $\mathcal{P}$ is defined through the formula

$$\mu(A) = \mathbb{E}\left[\int_{\mathbb{R}_+} \mathbf{1}_A \; d \; Trace\langle M, M\rangle\right].$$

This measure will serve as a control measure. The first step of the proof is to generalise the stability property.

**Lemma 3.2.** *Let $\mathbb{Q} \in \mathcal{S}^e$, suppose that $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(q \cdot M)$ and let $A \in \mathcal{P}$ be such that $\mathbb{E}\left[\mathcal{E}\left(q\mathbf{1}_A \cdot M\right)_\infty\right] = 1$, then we have that*

$$\mathcal{E}\left(q\mathbf{1}_A \cdot M\right) \in \mathcal{S}^e.$$

*Proof.* The proof follows from Proposition 2.1 as soon as the predictable set $A \in \mathcal{A}$. For the general case we take a sequence of predictable sets $A_n \in \mathcal{A}$ so that $\mu(A_n \Delta A) \to 0$. Of course we have that each element $\mathcal{E}(q\mathbf{1}_{A_n} \cdot M) \in \mathcal{S}^e$. The sequence $\mathcal{E}(q\mathbf{1}_{A_n} \cdot M)_\infty$ tends in probability to the element $\mathcal{E}(q\mathbf{1}_A \cdot M)_\infty$ and Scheffé's lemma implies that the convergence takes place in $\mathbf{L}^1$. Since $\mathcal{S}$ is closed we have that $\mathcal{E}(q\mathbf{1}_A \cdot M) \in \mathcal{S}$. But $\langle q\mathbf{1}_A \cdot M, q\mathbf{1}_A \cdot M\rangle_\infty \leq \langle q \cdot M, q \cdot M\rangle_\infty < \infty$ a.s. and therefore $\mathcal{E}(q\mathbf{1}_A \cdot M)_\infty > 0$ a.s. . Therefore $\mathcal{E}(q\mathbf{1}_A \cdot M) \in \mathcal{S}^e$. The proof of the lemma is complete. $\square$

*Remark.* It is not true that the stochastic exponential $\mathcal{E}(q\mathbf{1}_A \cdot M)$ is uniformly integrable as soon as the stochastic exponential $\mathcal{E}(q \cdot M)$ is uniformly integrable. The assumption $\mathbb{E}\left[\mathcal{E}\left(q\mathbf{1}_A \cdot M\right)_\infty\right] = 1$ cannot be omitted.

**Lemma 3.3.** *Let $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{S}^e$, suppose that $d\mathbb{Q}^{1,2}/d\mathbb{P} = \mathcal{E}(q^{1,2} \cdot M)$ and let $A \in \mathcal{P}$ be such that $\mathbb{E}\left[\mathcal{E}\left((q^1\mathbf{1}_A + q^2\mathbf{1}_{A^c}) \cdot M\right)_\infty\right] = 1$, then we have that*

$$\mathcal{E}\left((q^1\mathbf{1}_A + q^2\mathbf{1}_{A^c}) \cdot M\right) \in \mathcal{S}^e.$$

*Proof.* We omit the proof since it is almost a copy of the proof of the previous lemma. In fact we could have proved this lemma first. The previous lemma is then a special case.

The next step is to reduce our attention to elements of $\mathcal{S}$ that come from bounded integrands. More precisely, for each $\lambda > 0$ we introduce

$$\mathcal{S}^\lambda = \left\{\mathbb{Q} \in \mathcal{S} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(q \cdot M) \text{ and } \|q\| \leq \lambda\right\} = \left(\mathcal{S}^\lambda\right)^e.$$

The previously lemma allows us to prove the following density result

**Lemma 3.4.** *The sets $\mathcal{S}^\lambda$ are $m$–stable, form an increasing family and the union $\cup_{\lambda>0}\mathcal{S}^\lambda$ is $\mathbf{L}^1$–dense in $\mathcal{S}$.*

*Proof.* The stability of the sets $\mathcal{S}^\lambda$ is obvious from the definition of $m$–stability. That they are increasing in $\lambda$ is also obvious. That the sets are subsets of $\mathcal{S}$ follows from the previous lemma. Indeed if $\mathcal{E}(q \cdot M) \in \mathcal{S}$ then necessarily we must have that $\mathcal{E}(q\mathbf{1}_{\|q\|\leq\lambda} \cdot M) \in \mathcal{S}$. Indeed by Novikov's criterion (see [RY]), the stochastic exponential $\mathcal{E}(q\mathbf{1}_{\|q\|\leq\lambda} \cdot M)$ is uniformly integrable (remember that $Trace\langle M, M\rangle \leq 1$). For $\lambda \to \infty$ we also have that $\mathcal{E}(q\mathbf{1}_{\|q\|\leq\lambda} \cdot M)_\infty$ converges in probability to $\mathcal{E}(q \cdot M)_\infty$. Scheffé's lemma transforms the convergence into $\mathbf{L}^1$–convergence, proving the density. $\square$

**Lemma 3.5.** *The sets $\mathcal{S}^\lambda$ are convex and closed.*

*Proof.* The convexity will be checked using Itô's formula. So let us take

$$Z^1 = \mathcal{E}(q^1 \cdot M) \in \mathcal{S}^\lambda, \|q^1\| \leq \lambda \text{ and}$$
$$Z^2 = \mathcal{E}(q^2 \cdot M) \in \mathcal{S}^\lambda, \|q^2\| \leq \lambda.$$

Take $0 < \alpha < 1$. Itô's formula now gives

$$\begin{aligned} &d(\alpha Z^1 + (1-\alpha) Z^2)_t \\ &\quad = \alpha Z^1_t q^1_t \, dM_t + (1-\alpha) Z^2_t q^2_t \, dM_t \\ &\quad = \left(\alpha Z^1_t + (1-\alpha) Z^2_t\right) \left( \frac{\alpha Z^1_t q^1_t}{\alpha Z^1_t + (1-\alpha) Z^2_t} + \frac{(1-\alpha) Z^2_t q^2_t}{\alpha Z^1_t + (1-\alpha) Z^2_t} \right) dM_t \\ &\quad = q_t \, dM_t \qquad \text{where} \\ &q_t = \left( \frac{\alpha Z^1_t q^1_t}{\alpha Z^1_t + (1-\alpha) Z^2_t} + \frac{(1-\alpha) Z^2_t q^2_t}{\alpha Z^1_t + (1-\alpha) Z^2_t} \right). \end{aligned}$$

Obviously $\|q_t\| \leq \lambda$ since both $\|q^1_t\| \leq \lambda$ and $\|q^2_t\| \leq \lambda$. Because $\mathcal{S}$ is convex we have that $\mathcal{E}(q \cdot M)$ is already in $\mathcal{S}$. The boundedness on $q$ then implies $\mathcal{E}(q \cdot M) \in \mathcal{S}^\lambda$.

We still have to show that the set $\mathcal{S}^\lambda$ is closed. Let us take a sequence $\mathcal{E}(q^n \cdot M) \in \mathcal{S}^\lambda$ that converges in $\mathbf{L}^1$ to a martingale $Z$. Of course we suppose that $\|q^n\| \leq \lambda$ for all $n$. First observe that by taking the stochastic logarithm, we can easily see that the sequence $q^n \cdot M$ converges in the semi-martingale topology to a martingale $N$ and that $Z = \mathcal{E}(N)$. But by the uniform boundedness of the sequence $q^n$, we must then also have that the convergence takes place in all spaces $\mathbf{L}^p$. The sequence $q^n$ forms a bounded sequence in the space $\mathbf{L}^\infty(\mu)$ and therefore there is a sequence of convex combinations

$$k_n \in \; conv\{q^n, q^{n+1}, \dots\},$$

so that $k_n \to k$ in $\mu-$measure. Of course $k$ is predictable and $\|k\| \leq \lambda$. We also have that $k_n \cdot M$ converges in probability to $k \cdot M$ (even in all $\mathbf{L}^p$). Therefore we also have that $N = k \cdot M$. This shows that $Z$ is of the form $Z = \mathcal{E}(k \cdot M)$ where $k$ remains bounded by $\lambda$. This completes the proof of the lemma.

We now introduce the set

$$\mathcal{C}^\lambda = \left\{ q \colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^d \mid \|q\| \leq \lambda \text{ predictable and } \mathcal{E}(q \cdot M) \in \mathcal{S}^\lambda \right\}.$$

The following lemma seems an obvious consequence of the closedness of the sets $\mathcal{S}^\lambda$, so we omit the proof.

**Lemma 3.6.** *$\mathcal{C}^\lambda \subset \mathbf{L}^\infty(\mu)$ is closed for the topology of convergence in $\mu-$measure.*

The difficult part of the proof of the main result is to show that the sets $\mathcal{C}^\lambda$ are convex. The rather technical proof of this convexity result is based on the following BMO-style inequality.

**Lemma 3.7.** *Let $(\mathcal{G}_t)_t$ be a filtration satisfying the usual assumptions. Suppose that $V$ is a continuous martingale adapted to $\mathcal{G}$ and such that $V_0 = 0$. Suppose that $\langle V, V\rangle_\infty \le K$ for some constant $K$. Then a.s.*

$$\mathbb{E}\left[\left(\frac{\mathcal{E}(V)_\infty + \mathcal{E}(-V)_\infty}{2}\right)^2 \,\middle|\, \mathcal{G}_0\right] \le \cosh(K).$$

*Proof.* The proof uses the DDS-time change theorem, see [RY]. This theorem allows us to reduce the problem to the Brownian Motion case. Here are the details. First of all, remark that the martingale $V$ converges at $\infty$ and therefore we can close the interval $\mathbb{R}_+$ by adding the point $+\infty$. We then transform the interval $[0, +\infty]$ to the interval $[0, 1]$. After time one we continue the process by adding an independent Brownian Motion. The new process is still denoted by $V$ and the filtration is still denoted by $\mathcal{G}$, no confusion is possible. For this new process we define the finite stopping time

$$\tau = \inf\{t \mid \langle V, V\rangle_t > K\}.$$

By the assumption on the bracket of $V$, $\tau \ge 1$. From the DDS theorem it follows that $V_\tau$ is a random variable that has a gaussian distribution with mean 0 and variance $K$. However this random variable is independent of $\mathcal{G}_0$. By Jensen's inequality for conditional expectations, applied to the martingale $\frac{\mathcal{E}(V)+\mathcal{E}(-V)}{2}$, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\mathcal{E}(V)_1 + \mathcal{E}(-V)_1}{2}\right)^2 \,\middle|\, \mathcal{G}_0\right] \le \\
&\mathbb{E}\left[\left(\frac{\mathcal{E}(V)_\tau + \mathcal{E}(-V)_\tau}{2}\right)^2 \,\middle|\, \mathcal{G}_0\right] = \\
&\mathbb{E}\left[\left(\frac{\mathcal{E}(V)_\tau + \mathcal{E}(-V)_\tau}{2}\right)^2\right].
\end{aligned}$$

The latter quantity can easily be calculated and gives

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\mathcal{E}(V)_\tau + \mathcal{E}(-V)_\tau}{2}\right)^2\right] &= \frac{1}{4}e^{-K}\mathbb{E}\left[\exp(2V_\tau) + \exp(-2V_\tau) + 2\right] \\
&= \frac{1}{4}e^{-K}\left(e^{2K} + e^{2K} + 2\right) \\
&= \frac{1}{2}(e^K + e^{-K}) = \cosh(K).
\end{aligned}$$

□

**Lemma 3.8.** *Let the sequence of stopping times $\left(T_n^k\right)_{0\le k\le 2^n}$ be defined as follows. For each $n$ and $0 \le k \le 2^n$, we define:*

$$T_n^k = \inf\left\{t \mid \langle M, M\rangle_t \ge \frac{k}{2^n}\right\}.$$

*Obviously* $T_n^0 = 0$ *and* $T_n^{2^n} = \infty$ *since* $\langle M, M\rangle_\infty < 1$. *Let* $q^1$ *and* $q^2$ *be predictable* $\mathbb{R}^d$ *valued processes bounded by* $\lambda$. *For each* $n$ *we define*

$$f_n = \prod_{k=0}^{2^n-1} \left( \frac{1}{2}\mathcal{E}\left(\mathbf{1}_{]T_n^k, T_n^{k+1}]} q^1 \cdot M\right)_\infty + \frac{1}{2}\mathcal{E}\left(\mathbf{1}_{]T_n^k, T_n^{k+1}]} q^2 \cdot M\right)_\infty \right).$$

*Let*

$$f = \mathcal{E}\left(\frac{q^1 + q^2}{2} \cdot M\right)_\infty$$

*Then* $f_n$ *tends to* $f$ *in* $\mathbf{L}^1(\mathbb{P})$.

*Proof.* Clearly $f > 0$ and $\mathbb{E}_\mathbb{P}[f] = 1$. Define the measure $\mathbb{Q}$ as $d\mathbb{Q} = f\, d\mathbb{P}$. We will show that

$$\|f_n - f\|_{\mathbf{L}^1(\mathbb{P})} = \|\frac{f_n}{f} - 1\|_{\mathbf{L}^1(\mathbb{Q})}$$

tends to zero. Obviously $\mathbb{E}_\mathbb{Q}\left[\frac{f_n}{f}\right] = 1$. The statement therefore follows as soon as we can prove that $\mathbb{E}_\mathbb{Q}\left[\left(\frac{f_n}{f}\right)^2\right] \to 1$. Indeed this convergence immediately implies that $\|\frac{f_n}{f} - 1\|^2_{\mathbf{L}^2(\mathbb{Q})} \to 0$.
Under the measure $\mathbb{Q}$, the martingale $M$ can be decomposed into a martingale $N$ and a process of finite variation. The continuous martingale $N$ has the same bracket as $M$. Moreover a straightforward calculation shows that each factor, say $g_n^k$ in the expression of $\frac{f_n}{f}$ can be written as

$$g_n^k = \frac{1}{2}\mathcal{E}\left(\mathbf{1}_{]T_n^k, T_n^{k+1}]} \frac{q^1 - q^2}{2} \cdot N\right)_\infty + \frac{1}{2}\mathcal{E}\left(\mathbf{1}_{]T_n^k, T_n^{k+1}]} \frac{q^2 - q^1}{2} \cdot N\right)_\infty .$$

We now repeatedly will use the lemma 3.8. The bracket we must control is

$$\langle \mathbf{1}_{]T_n^k, T_n^{k+1}]} \frac{q^1 - q^2}{2} \cdot N, \mathbf{1}_{]T_n^k, T_n^{k+1}]} \frac{q^1 - q^2}{2} \cdot N\rangle_\infty \leq \lambda^2 2^{-n}.$$

By telescoping and repeated application of lemma 3.8, we now find

$$\begin{aligned}
\mathbb{E}_\mathbb{Q}\left[\prod_{k=0}^{2^n-1} (g_n^k)^2\right] &\leq \mathbb{E}_\mathbb{Q}\left[\prod_{k=0}^{2^n-2} (g_n^k)^2 \, \mathbb{E}_\mathbb{Q}\left[\left(g_n^{2^n-1}\right)^2 \mid \mathcal{F}_{T_n^{2^n-1}}\right]\right] \\
&\leq \mathbb{E}_\mathbb{Q}\left[\prod_{k=0}^{2^n-2} (g_n^k)^2 \, \cosh(\lambda^2 2^{-n})\right] \\
&\leq \dots \\
&\leq \left(\cosh(\lambda^2 2^{-n})\right)^{2^n}
\end{aligned}$$

Since for small $x$ we have $\cosh(x) \approx 1 + x^2/2$, we get that $\left(\cosh(\lambda^2 2^{-n})\right)^{2^n}$ tends to 1 as $n$ tends to infinity. $\square$

**Lemma 3.9.** *The set $\mathcal{C}^\lambda$ is convex.*

*Proof.* We use the notation of the previous lemma. Obviously we have that $f_n \in \mathcal{S}^\lambda$, therefore also $f \in \mathcal{S}^\lambda$ and therefore $\frac{q^1+q^2}{2} \in \mathcal{C}^\lambda$. Since $\mathcal{C}^\lambda$ is already closed for convergence in probability, we get that convex combinations with other coefficients than 1/2 remain also in $\mathcal{C}^\lambda$. □

The next step in the proof is to analyse the structure of the set $\mathcal{C}^\lambda$. The following lemma is obvious in the sense that either the statements were proved before or are trivial

**Lemma 3.10.** *The set $\mathcal{C}^\lambda$ satisfies the following properties*

(1) $\mathcal{C}^\lambda \subset \mathbf{L}^\infty(\mathcal{P}, \mu; \mathbb{R}^d)$
(2) *$\mathcal{C}^\lambda$ is contained in the ball of radius $\lambda$*
(3) *$\mathcal{C}^\lambda$ is closed for $\mu$ convergence*
(4) *$\mathcal{C}^\lambda$ is convex, therefore it is also weak*, i.e. $\sigma(\mathbf{L}^\infty(\mathcal{P}, \mu; \mathbb{R}^d), \mathbf{L}^1(\mathcal{P}, \mu; \mathbb{R}^d))$ compact.*
(5) *if $q^1, q^2 \in \mathcal{C}^\lambda$, if $A \in \mathcal{P}$, then $q^1\mathbf{1}_A + q^2\mathbf{1}_{A^c} \in \mathcal{C}^\lambda$*

The following lemmma can be seen as an extension of the convexity property. This property is sometimes called, predictably convex.

**Lemma 3.11.** *If $q^1, q^2 \in \mathcal{C}^\lambda$ if $h$ is real valued predictable process such that $0 \le h \le 1$, then also $h\, q^1 + (1-h)\, q^2 \in \mathcal{C}^\lambda$.*

*Proof.* If $h = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$ where $A_i \in \mathcal{P}$ and where the nonnegative numbers $\alpha_i$ sum up to 1, the property follows from convexity. However the set of such convex combinations is dense (for the topology in $\mu-$convergence) in the set of all functions between 0 and 1. The closedness property completes the argument. □

We are now ready to prove the theorem for the set $\mathcal{S}^\lambda$.

**Theorem 3.12.** *With the notation of above, there exists a compact convex set-valued function $\Phi^\lambda : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ so that*

(1) *The graph of $\Phi^\lambda$ is in $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$*
(2) *$0 \in \Phi^\lambda(t, \omega)$ for each $(t, \omega)$.*
(3) $\mathcal{C}^\lambda = \{q : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d \mid q \textit{ is predictable and } \mu \textit{ a.s. } q(t,\omega) \in \Phi^\lambda(t,\omega)\}$

*Proof.* For each $A \in \mathcal{P}$ we define the set $C(A)$ as follows

$$C(A) = \left\{ \int_A q\, d\mu \mid q \in \mathcal{C}^\lambda \right\}.$$

The set-valued mapping $C : \mathcal{P} \to \mathbb{R}^d$ satisfies

(1) $0 \in C(A)$ since $0 \in \mathcal{C}^\lambda$
(2) if $x \in C(A)$ then $\|x\| \le \lambda\mu(A)$. Indeed $\| \int_A q\, d\mu\| \le \int_A \|q\|\, d\mu \le \lambda\mu(A)$.
(3) $C(A)$ is convex since $\mathcal{C}^\lambda$ is convex.
(4) $C(A)$ is compact since $\mathcal{C}^\lambda$ is weak* compact.
(5) if $(A_n)_n$ is a sequence of pairwise disjoint predictable sets with $A = \cup_N A_n$, if $x_n = \int_{A_n} q^n\, d\mu \in C(A_n)$ then the sum $x = \sum_n x_n$ converges and $x \in C(A)$. Indeed the convergence follows from the bound under item 2 and $x = \int_A q\, d\mu$ where $q = \sum_n \mathbf{1}_{A_n} q^n$. The latter follows from Lebesgue's dominated convergence theorem and the closedness of $\mathcal{C}^\lambda$.

The theorem of Debreu-Schmeidler, [DeS] now gives the existence of a compact convex set-valued, $\mathcal{P}\otimes(\mathbb{R}^d)$ measurable mapping $\Phi^\lambda$ so that for all $A$ we have

$$C(A)=\left\{\int_A q\,d\mu \;\mid\; q\in\Phi^\lambda \;\mu \text{ a.s.}\right\}.$$

We still have to show that this allows to find the set $\mathcal{C}^\lambda$.

Let us first suppose that $q\in\mathcal{C}^\lambda$. We have to show that $\mu$ a.s. we have that $q\in\Phi^\lambda$. In case this were false the set

$$A=\{(t,\omega)\mid q(t,\omega)\notin\Phi^\lambda(t,\omega)\}$$

is measurable and has a positive measure $\mu(A)>0$. Because of the density of the points with rational coordinates and the convexity of the sets $\Phi^\lambda$, this means that there is a vector $p\in\mathbb{R}^d$ (with rational coordinates) such that the set

$$A_p=\{(t,\omega)\mid\langle p,q(t,\omega)\rangle>\sup\langle p,\Phi^\lambda(t,\omega)\rangle\}$$

also has a positive measure $\mu(A_p)>0$. Indeed by the separation theorem we can write that

$$A=\cup_{p\in\mathbb{R}^d,p\text{ rational}}A_p.$$

But then necessarily we have that $\int_{A_p}q\,d\mu\notin C(A_p)$, since obviously we have that $\langle p,\int_{A_p}q\,d\mu\rangle=\int_{A_p}\langle p,q\rangle\,d\mu>\int_{A_p}\sup\langle p,\Phi^\lambda(t,\omega)\rangle\,d\mu\ge\sup_{g\in\Phi^\lambda}\int_{A_p}\langle p,g\rangle\,d\mu=\sup\langle p,C(A_p)\rangle$.

The converse is proved in a similar way. So let $q_0$ be a predictable selector of $\Phi^\lambda$, we have to show that $q_0\in\mathcal{C}^\lambda$. If this is not the case then we separate the point $q_0$ from the compact convex set $\mathcal{C}^\lambda$. This we can do by the Hahn-Banach theorem. We obtain a function $f\in\mathbf{L}^1(\mu;\mathbb{R}^d)$ so that

$$\int\langle f,q_0\rangle\,d\mu>\sup_{q\in\mathcal{C}^\lambda}\int\langle f,q\rangle\,d\mu.$$

The sup is actually attained because of compactness of $\mathcal{C}^\lambda$. Let a maximising element be $q_1$. But then we must necessarily have that for every $q\in\mathcal{C}^\lambda$ the inequality $\langle f,q_1\rangle\ge\langle f,q\rangle$ holds $\mu$ a.s. . This follows from the property 5 of lemma 3.10. Indeed if there would be an element $q\in\mathcal{C}^\lambda$ so that the set $B=\{\langle f,q_1\rangle<\langle f,q\rangle\}$ is not negligable, we could replace $q_1$ by $q_1\mathbf{1}_{B^c}+q\mathbf{1}_B$ yielding a greater expression than the one for $q_1$.

The set $\{\langle f,q_0\rangle>\langle f,q_1\rangle\}$ must have a strictly positive $\mu$ measure. Since the simple functions are dense in $\mathbf{L}^1$ we can find a vector $p\in\mathbb{R}^d$ as well as an $\epsilon>0$ so that the set $A_p=\{\langle f,q_0\rangle>\epsilon+\langle f,q_1\rangle\}\cap\{\|f-p\|_{\mathbb{R}^d}\le\frac{\epsilon}{4\lambda}\}$ has a nonzero measure, $\mu(A_p)>0$.

But this inequality and the fact that all the functions in $\mathcal{C}^\lambda$ are pointwise bounded by $\lambda$, implies that

$$\begin{aligned}\langle p, \int_{A_p} q_0\, d\mu\rangle &\geq \int_{A_p} \langle f, q_0\rangle\, d\mu - \frac{\epsilon}{4}\mathbb{P}[A_p]\\ &> \int_{A_p} \langle f, q_1\rangle\, d\mu + \frac{3\epsilon}{4}\mathbb{P}[A_p]\\ &\geq \sup_{q\in\mathcal{C}^\lambda}\int_{A_p} \langle f, q\rangle\, d\mu + \frac{3\epsilon}{4}\mathbb{P}[A_p]\\ &\geq \sup_{q\in\mathcal{C}^\lambda}\int_{A_p} \langle p, q\rangle\, d\mu + \frac{\epsilon}{2}\mathbb{P}[A_p].\end{aligned}$$

And therefore we must have that $\int_{A_p} q_0\, d\mu \notin C(A_p)$. But this is a contradiction to $q_0 \in \Phi^\lambda$ and the definition of the Radon-Nikodym derivative for set-valued measures. $\square$

*Remark.* The proof was based on the Radon-Nikodym theorem for set-valued measures. However the proof of the version we need can be given using the support functionals. Since we also needed the support functionals in a later stage of the proof there is a shortcut in the sense of merging the proof of the RN-theorem together with the arguments on the support functionals. However this would have obscured the idea of the proof.

The next step consists in getting rid of the truncation.

**Lemma 1.13.** *If $\lambda \leq \nu$ then $\Phi^\lambda = \Phi^\nu \cap B_\lambda$, where $B_\lambda$ denotes the ball of radius $\lambda$ in the Euclidean space $\mathbb{R}^d$. As a consequence we have $\Phi^\lambda \subset \Phi^\nu$ $\mu$ a.s. .*

*Proof.* If this is not the case we will make a measurable selection, say $q$, of the set-valued function $\Phi^\lambda \setminus (\Phi^\nu \cap B_\lambda)$, at least on the predictable set $A$, where $\Phi^\lambda = \setminus (\Phi^\nu \cap B_\lambda)$ is nonempty. Let us put $q$ equal to 0 where the set is empty. Since $q$ is in $\mathcal{C}^\lambda$ it has to be in $\mathcal{C}^\nu$ as well. Since obviously the element $q = q\mathbf{1}_A$ is also in $\mathcal{C}^\nu$ it has to be a selector of $\Phi^n u$. But this is a contradiction to the construction of $q$. The converse inclusion is proved in the same way. $\square$

For each $(t,\omega)$ we now define

$$\Phi(t,\omega) = \cup_{\lambda>0}\Phi^\lambda(t,\omega) = \cup_{n\geq 1}\Phi^n(t,\omega).$$

Because $\Phi$ is the union of an increasing sequence of convex sets, it is convex. The closedness of $\Phi$ is something that needs a proof, since the countable union of closed sets does not have to be closed. But since we have the equality $\Phi^\lambda = \Phi^\nu \cap B_\lambda$, the union is indeed closed.

The last part consists in showing that we get the m–stable set $\mathcal{S}$ back.

**Lemma 3.14.** *Let $q \in \Phi$, $\mu$ a.s. . Suppose that $\mathbb{E}\left[(\mathcal{E}(q\cdot M))_\infty\right] = 1$, meaning that $\mathcal{E}(q\cdot M)$ is a uniformly integrable martingale. Then $\mathcal{E}(q\cdot M) \in \mathcal{S}$*

*Proof.* By construction we have that $q_n = q\mathbf{1}_{\|q\|\leq n}$ is a selector of $\Phi^n$. Therefore it is in $\mathcal{C}^n$. But then we have that $\mathcal{E}(q_n\cdot M)$ is in $\mathcal{S}$. Since $\mathcal{E}(q_n\cdot M)$ tends to $\mathcal{E}(q\cdot M)$ in $\mathbf{L}^1$ (by Scheffé's lemma), we must have that $\mathcal{E}(q\cdot M) \in \mathcal{S}$. $\square$

The following statement concludes the proof of theorem 3.1 (or 1.4).

**Theorem 3.15.** *We have the following equality*

$$\mathcal{S} = \{\mathcal{E}(q \cdot M) \mid q \in \Phi \text{ and } \mathbb{E}\left[(\mathcal{E}(q \cdot M))_\infty\right] = 1\}$$

*Proof.* One inclusion is in the previous lemma. The other inclusion is quite obvious. Take $q$ so that $\mathcal{E}(q \cdot M)$ is in $\mathcal{S}$. Obviously lemma 3.2 implies that for each $n$ we have that $q\mathbf{1}_{\|q\| \le n}$ is in $\mathcal{C}^n$. Therefore we have that $q\mathbf{1}_{\|q\| \le n} \in \Phi^n \subset \Phi$, $\mu$ a.s. . This of course implies that $q \in \Phi$, $\mu$ a.s. . $\square$

*Remark.* Kabanov pointed out that the proof of the above theorem can be simplified when the following reformulation of his result on extreme points is used, see [Kab]. This is also the topic of ongoing research.

**Theorem 3.16.** *We use the notation of theorem 3.1 and we suppose that for each $(t, \omega)$ the set $C(t, \omega)$ is bounded and convex. If $\partial$ denotes the operator that associates with a set $S$ the set of its extreme points $\partial S$, we have that*

$$\partial\mathcal{S} = \{\mathcal{E}(q \cdot M)_\infty \mid q \text{ predictable }, q \in \partial C \text{ and } \mathbb{E}_\mathbb{P}\left[\mathcal{E}(q \cdot M)_\infty\right] = 1\}.$$

## 4. The m–stable hull and the relation with some riskmeasures

In this section we will investigate if the classical examples of risk measures ([De]) come from m–stable sets. We start with the obvious

**Lemma 4.1.** *If $\mathcal{S}$ is a set of probability measures $\mathcal{S} \subset \mathbf{L}^1$, then the intersection of all m–stable, convex, closed sets containing $\mathcal{S}$ is still an m–stable, closed, convex set. It is the smallest closed, convex, m–stable set containing $\mathcal{S}$ and is called the m–stable hull of $\mathcal{S}$.*

*Proof.* This follows immediately from the definition of m–stable sets.

The following theorem deals with the case of Tailvar or CV@R. For the definition see [De]. Just for the information of the reader, let us recall that the sigma-algebra $\mathcal{F}_0$ is trivial and therefore that for every element $Z \in \mathcal{S}$ we have that $Z_0 = 1$.

**Theorem 4.2.** *Suppose that all martingales for the filtration $\mathcal{F}$ are continuous (as well known, this is equivalent to the property that all stopping times are predictable). Suppose that $K > 1$ and let*

$$\mathcal{S}_K = \left\{\mathbb{Q} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \le K\right\}.$$

*Then the m–stable hull of $\mathcal{S}_K$ is the set of all probability measures absolutely continuous with respect to $\mathbb{P}$.*

*Proof.* We will show that every probability measure $\mathbb{Q}$ can be approximated by probability measures that are concatenations of elements of $\mathcal{S}_K$. Since the measures $\mathbb{Q}$ with densities that are bounded away from zero form a dense set, it does not harm to suppose that $\mathbb{Q}$ has a density process

$$Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$$

satisfying $0 < \epsilon \le Z$. By hypothesis the martingale $Z$ is continuous. Let us now define a sequence of stopping times starting with $T_0 = 0$ and inductively defined as

$$T_k = \inf \left\{ u \mid u > T_{k-1} \text{ and } \frac{Z_u}{Z_{T_{k-1}}} \ge K \right\}.$$

Because of the continuity we have that the random variables

$$f_k = \frac{Z_{T_k}}{Z_{T_{k-1}}}$$

are bounded by $K$ and are therefore densities of elements in $\mathcal{S}_K$. Furthermore, because the intervals $]\!]T_{k-1}, T_k]\!]$ are disjoint, the products $Z_{T_N} = \prod_{k=1}^N f_k$ are concatenations of elements of $\mathcal{S}_K$ and therefore they are densities of probability measures which, by Proposition 2.1, are necessarily in the m–stable hull of $\mathcal{S}_K$. Because the martingale $Z$ is continuous and converges at $t = \infty$ (at this point we only need that it has left limits for each $t \le \infty$), we must have that $\mathbb{P}[T_N = \infty]$ tends to 1. Since $Z$ is a uniformly integrable martingale, the convergence of $Z_{T_N}$ to $Z_\infty$ is both a.s. and in $\mathbf{L}^1$. This shows that $\mathbb{Q}$ is in the m–stable hull of $\mathcal{S}_K$. □

For law-invariant risk measures we can prove the following

**Theorem 4.3.** *Suppose that the filtration $\mathcal{F}$ is generated by a $d-$dimensional Brownian Motion $W$. If $\mathcal{S}$ is an m–stable convex closed set, if $\mathcal{S}$ is law invariant, then either $\mathcal{S} = \{\mathbb{P}\}$ or $\mathcal{S}$ equals the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$.*

*Proof.* The proof is based on the following

**Lemma 4.4 (Skorohod stopping problem).** *If $B$ is a Brownian Motion (with respect to a filtration $\mathcal{G}$). If $\nu$ is a probability on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} x\, \nu(dx) = 1$, then there is a ($\mathcal{G}$)-stopping time $\tau$ such that $\mathcal{E}(B)^\tau$ is uniformly integrable and $\mathcal{E}(B)_\tau$ has as its law, the probability $\nu$.*

*Proof of the lemma.* The proof is almost identical to the proof of the usual stopping time problem, see [RY] and the references given there. For completeness we present an easy proof. Let $\mathcal{R}^n$ be an increasing sequence of finite $\sigma-$algebras on $\mathbb{R}_+$, chosen such that they generate the Borel $\sigma-$algebra and such that each atom of $\mathcal{R}^n$ is split into exactly two atoms of $\mathcal{R}^{n+1}$. For convenience we take $\mathcal{R}^0 = \{\emptyset, \mathbb{R}_+\}$. Define the $\mathcal{R}^n$ measurable, conditional expectation $y_n = \mathbb{E}_\nu \left[ id_{\mathbb{R}_+} \mid \mathcal{R}^n \right]$. Inductively we define an increasing sequence of stopping times $\sigma_n$ so that $\mathcal{E}(B)_{\sigma_n}$ has the same law as $y_n$. For $n = 0$ we take $\sigma_0 = 0$. Then $y_1$ takes two values (at most) and we take $\sigma_1$ as the first time that $\mathcal{E}(B)_t$ takes one of these values. Clearly $\mathcal{E}(B)_{\sigma_1}$ takes the same two values (at most). On each of the atoms, generated by the random variable $\mathcal{E}(B)_{\sigma_1}$, we define $\sigma_2$ as the first time after $\sigma_1$, where $\mathcal{E}(B)$ takes one of the (at most two) corresponding values of $y_2$. We continue this procedure and we obtain a martingale (not only a localmartingale) $(\mathcal{E}(B)_{\sigma_n})_n$. The sequence $\sigma_n$ can clearly be defined since the only difficulty is when $\sigma_n = \infty$ with some probability, in which case we have $\mathcal{E}(B)_{\sigma_n}$ takes the value zero. When this is the case, the martingale $y_n = 0$ on the corresponding set and consequently the further values of $y_k$ are all zero as well. From the construction it follows that for each $n$, $\mathcal{E}(B)_{\sigma_n}$ has the same law as $y_n$. Indeed since there are at most two values, the probabilities of these values are determined by the fact that their average is the preceding value of the martingale. These are the same for the sequence $(y_n)_n$ as for the sequence $(\mathcal{E}(B)_{\sigma_n})_n$. Let us now define $\tau = \lim \sigma_n$. Since the law of $\mathcal{E}(B)_\tau$ is the limit of the laws of $\mathcal{E}(B)_{\sigma_n}$, this law is precisely the limit law of $y_n$, hence $\nu$. This implies that the process $\mathcal{E}(B)^\tau$ is a uniformly integrable martingale since obviously it is a nonnegative local martingale, starting at 0 and $\mathbb{E}[\mathcal{E}(B)_\tau] = \int_{\mathbb{R}_+} x\, \nu(dx) = 1$. $\square$

We can now continue the proof of the theorem. We suppose that $\mathcal{S}$ contains an element $f > 0$, a.s. , that is different from 1. The law of $f$ will be denoted by $\nu$. We have to show that $\mathcal{S}$ equals the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$. What we will do is show that the set $C$, from the representation theorem 3.1 is equal to $\mathbb{R}^d$. Take a vector $x \in \mathbb{R}^d$ of unit length. Let us consider the process $B = x.W$. This process is clearly a $1-$dimensional Brownian Motion for the filtration $\mathcal{F}$. Let $\tau_1$ be a stopping time such that $\mathcal{E}(B)_{\tau_1}$ has the law $\nu$. By Skorohod's theorem (see the lemma above), this is possible. Since $f > 0$ a.s. , we must have that $\tau_1 < +\infty$. The stopping time $\tau_1$ can be taken to be a stopping time with respect to the filtration generated by $B$. Because of Blumenthal's zero-one law (see [RY]) we must have that $\tau_1 > 0$ a.s. . The process $q = x\mathbf{1}_{[\![0,\tau_1]\!]}$ is therefore a selector of the set $C$. If we restart the Brownian Motion $B$ at time $\tau$, i.e. if we look at the process $B'_s = 0$ for $s \le \tau$ and $B'_s = B_{\tau+s} - B_\tau$ we can again apply Skorohod's theorem and we get a second stopping time $\tau_2 > \tau_1$ such that $\mathcal{E}(B')_{\tau_2}$ has the same law $\nu$. Moreover the random variable $\mathcal{E}(B')_{\tau_2}$ is independent of $\mathcal{E}(B)_{\tau_1}$. This means that also the process $q = x\mathbf{1}_{]\!]\tau_1,\tau_2]\!]}$ is a selector of $C$. If we continue in the same way, we get a strictly increasing sequence of stopping times $\tau_n$ such that for each $k$ we have that $x\mathbf{1}_{]\!]\tau_{k-1},\tau_k]\!]}$ is a selector of $C$ and such that the random variables $f_k = \mathcal{E}(\mathbf{1}_{]\!]\tau_{k-1},\tau_k]\!]} \cdot B)_{\tau_k}$ are iid with law $\nu$. Since as easily seen, the product $\Pi_0^\infty f_k$ diverges to zero a.s. , we must have that the sequence $\tau_k$ tends to $\infty$. It follows that $x \in C$ on $\mathbb{R}_+ \times \Omega$. We now apply the same reasoning to the process $B" = nB = nx.W$. Since $B"$ is, up to scaling by $\sqrt{n}$, a Brownian Motion we can find a stopping time $\tau"$ such that $\mathcal{E}(B")_{\tau"}$ has the law $\nu$ and $0 < \tau" < +\infty$. This means that $q = nx\mathbf{1}[\![0, \tau"]\!]$ is a selector of $C$. But the same reasoning as above, meaning that we restart at time $\tau"$ then gives that the process $nx$ is a selector of $C$. Since $C$ is convex and since for each vector $z \in \mathbb{R}^d$ we now have that $z \in C$ a.s. on $\mathbb{R}_+ \times \Omega$, we must have that $C = \mathbb{R}^d$, a.s. . $\square$

*Remark.* The proof can easily be adapted to the case where $M$ is a $d-$dimensional continuous local martingale with the predictable representation property and with the condition that for each coordinate $k$ we have that $\langle M^k, M^k\rangle_\infty = +\infty$. Although we have some generalisations of this situation but we do not know whether the theorem holds in the general case of a filtration where all martingales are continuous.

## 5. The construction of the Snell envelope

In this section we will assume without further notice that the set $\mathcal{S}$ is m–stable. The properties of m–stable sets allow us to define a risk adjusted value as a process. The construction goes as follows. For an m–stable set of probability measures $\mathcal{S}$ and for a bounded random variable $f \in \mathbf{L}^\infty$ and under the assumption that $\mathcal{F}_0$ is degenerate, we define the risk adjusted value at time zero, $\Phi_0(X)$ as

$$\Phi_0(f) = \inf \{\mathbb{E}_\mathbb{Q}[f] \mid \mathbb{Q} \in \mathcal{S}\}.$$

At intermediate times $0 \le t \le \infty$, we could try to define the random variable

$$\Phi_t(f) = \text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_t] \mid \mathbb{Q} \in \mathcal{S}\right\}.$$

For $t = 0$ there is no ambiguity in the definitions since the sigma-algebra $\mathcal{F}_0$ is trivial. The infimum is an infimum of random variables and therefore it has to be seen as an essential infimum. But the measures $\mathbb{Q} \in \mathcal{S}$ are not all equivalent to $\mathbb{P}$ and hence the conditional expectations are not defined $\mathbb{P}$ a.s. . However the random variable $\Phi_t(f)$ can also be defined in another way, thereby avoiding this difficulty. One way is to observe that the measures in $\mathcal{S}$, equivalent to $\mathbb{P}$, are dense in $\mathcal{S}$. Therefore we can define

$$\Phi_t(f) = \text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_t] \mid \mathbb{Q} \in \mathcal{S}, \mathbb{Q} \sim \mathbb{P}\right\}.$$

By the density argument we have that for *every* $\mathbb{Q}_0 \in \mathcal{S}$:

$$\text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_t] \mid \mathbb{Q} \in \mathcal{S}, \mathbb{Q} \sim \mathbb{P}\right\} \le \mathbb{E}_{\mathbb{Q}_0}[f \mid \mathcal{F}_t] \qquad \mathbb{Q}_0 \text{ a.s. .}$$

Another solution is to take

$$\Phi_t(f) = \text{ess.sup}\left\{g \mid \forall \mathbb{Q} \in \mathcal{S} : \ g \le \mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_t], \ \mathbb{Q} \text{ a.s. } \right\}.$$

The following theorem deals with risk adjusted values of a stochastic process. We will use a different but similar notation as the one introduced in the beginning of this section.

**Theorem 5.1.** *If $\mathcal{S}$ is an m–stable set and if $X$ is a bounded càdlàg adapted stochastic process, then there is a càdlàg process, denoted by $\Psi_t(X)$, so that for every stopping time $0 \le T < \infty$ we have*

$$\Psi_T(X) = \text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T] \mid \tau \ge T \textit{ is a stopping time and } \mathbb{Q} \in \mathcal{S}\right\}.$$

*We call the process $\Psi_t(X)$, the risk adjusted process corresponding to the process $X$. The process $\Psi(X)$ is a $\mathbb{Q}$–submartingale for every $\mathbb{Q} \in \mathcal{S}$.*

*Remark.* The proof follows the same lines as the proof of the existence of the Snell envelope, see [DM] pages 431 up to 436. Since we have the extra difficulty that we have to deal with all the measures in $\mathcal{S}$, we prefer to give the details. The reader can skip the proof.

*Remark.* That the process $\Psi(X)$ is a submartingale for every "test-probability" has a direct interpretation. It shows that as time evolves the uncertainty on the remaining part decreases. The risk adjusted value therefore increases in expected value.

*Proof.* We start with the definition of a family of random variables, indexed by the set of all stopping times $0 \le T \le \infty$:

$$Y_T = \text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T] \mid \tau \ge T \text{ is a stopping time and } \mathbb{Q} \in \mathcal{S}\right\}.$$

We emphasize that this is a only a family of random variables and that for the moment there is no process $Y$ that gives the values $Y_T$ at times $T$. The construction of such a process involves the selection of representatives of the a.s. equivalence classes $Y_T$.

**Lemma 5.2.** *For a fixed stopping time $T$, the set*

$$\{\mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T] \mid \ \tau \geq T \text{ is a stopping time and } \mathbb{Q} \in \mathcal{S}\}$$

*is a lattice.*

*Proof.* Indeed for $\tau_1, \tau_2 \geq T$ and $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{S}^e$, (with density processes $Z^1, Z^2$ resp), we have that

$$\min\left(\mathbb{E}_{\mathbb{Q}^1}[X_{\tau_1} \mid \mathcal{F}_T], \mathbb{E}_{\mathbb{Q}^2}[X_{\tau_2} \mid \mathcal{F}_T]\right) = \mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T],$$

where the measure $\mathbb{Q}$ is defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{Z^1_\infty}{Z^1_T} \text{ on the set } \{\mathbb{E}_{\mathbb{Q}^1}[X_{\tau_1} \mid \mathcal{F}_T] < \mathbb{E}_{\mathbb{Q}^2}[X_{\tau_2} \mid \mathcal{F}_T]\}$$

$$= \frac{Z^2_\infty}{Z^2_T} \text{ on the set } \{\mathbb{E}_{\mathbb{Q}^1}[X_{\tau_1} \mid \mathcal{F}_T] \geq \mathbb{E}_{\mathbb{Q}^2}[X_{\tau_2} \mid \mathcal{F}_T]\}.$$

That the measure $\mathbb{Q}$ is still in $\mathcal{S}$ follows from the m-stability of the set $\mathcal{S}$.
In the same way we define

$$\tau = \tau_1 \text{ on the set } \{\mathbb{E}_{\mathbb{Q}^1}[X_{\tau_1} \mid \mathcal{F}_T] < \mathbb{E}_{\mathbb{Q}^2}[X_{\tau_2} \mid \mathcal{F}_T]\}$$

$$= \tau_2 \text{ on the set } \{\mathbb{E}_{\mathbb{Q}^1}[X_{\tau_1} \mid \mathcal{F}_T] \geq \mathbb{E}_{\mathbb{Q}^2}[X_{\tau_2} \mid \mathcal{F}_T]\}. \quad \square$$

**Corollary 5.3.** *Because of this lattice property we also have that for every stopping time $0 \leq T < \infty$ and for every probability measure $\mu \ll \mathbb{P}$ (not necessarily in $\mathcal{S}$) that*

$$\mathbb{E}_\mu[Y_T] = \inf\left\{\mathbb{E}_\mu\left[\mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T]\right] \mid \mathbb{Q} \sim \mathbb{P}; \mathbb{Q} \in \mathcal{S}; T \leq \tau \text{ stopping time}\right\}.$$

*Proof.* Obvious.

**Lemma 5.4.** *The m-stability of the set $\mathcal{S}$ implies the equality of the following two sets: (here $\infty > \nu \geq \sigma$ is another stopping time)*

$$\left\{\left(\frac{Z_\nu}{Z_\sigma}, \frac{Z_\sigma}{Z_\tau}\right) \mid Z \in \mathcal{S}^e\right\} = \left\{\left(\frac{Z'_\nu}{Z'_\sigma}, \frac{Z_\sigma}{Z_\tau}\right) \mid Z \in \mathcal{S}^e, Z' \in \mathcal{S}^e\right\}.$$

*Proof.* The proof is obvious.

**Lemma 5.5.** *For every pair of stopping times $0 \leq \tau \leq \sigma < \infty$, we have that*

$$Y_\tau \leq \text{ess.inf}\left\{\mathbb{E}_{\mathbb{Q}}[Y_\sigma \mid \mathcal{F}_\tau] \ \mid \mathbb{Q} \in \mathcal{S}^e\right\}.$$

*Proof.* From the lattice property and the previous lemma, we can easily justify the following calculations

$$\begin{aligned}
Y_\tau &\leq \mathbb{E}\left[X_\nu \frac{Z_\nu}{Z_\tau} \mid \mathcal{F}_\tau\right] \text{ all } Z \in \mathcal{S}^e, \text{ all } \nu \geq \tau \\
&\leq \mathbb{E}\left[\mathbb{E}\left[X_\nu \frac{Z_\nu}{Z_\sigma} \mid \mathcal{F}_\sigma\right] \frac{Z_\sigma}{Z_\tau} \mid \mathcal{F}_\tau\right] \text{ all } Z \in \mathcal{S}^e, \text{ all } \nu \geq \sigma \\
&\leq \mathbb{E}\left[\mathbb{E}\left[X_\nu \frac{Z'_\nu}{Z'_\sigma} \mid \mathcal{F}_\sigma\right] \frac{Z_\sigma}{Z_\tau} \mid \mathcal{F}_\tau\right] \text{ all } Z \in \mathcal{S}^e, \text{ all } Z' \in \mathcal{S}^e, \text{ all } \nu \geq \sigma \\
&\leq \mathbb{E}\left[Y_\sigma \frac{Z_\sigma}{Z_\tau} \mid \mathcal{F}_\tau\right] \text{ all } Z \in \mathcal{S}^e, \text{ all } \nu \geq \sigma.
\end{aligned}$$

Taking the ess.inf over all $Z \in \mathcal{S}^e$ implies the desired inequality. $\square$

**Lemma 5.6.** *The family $Y_T$ constructed above satisfies the following submartingale property: for every pair of stopping times $0 \le \tau \le \sigma < \infty$ and every $\mathbb{Q} \in \mathcal{S}$ we have*

$$Y_\tau \le \mathbb{E}_\mathbb{Q}[Y_\sigma \mid \mathcal{F}_\tau].$$

*In particular it satisfies this property for $\mathbb{P}$.*

*Proof.* Follows directly from the previous lemma. □

**Lemma 5.7.** *The family $Y_T$ constructed above satisfies the following right continuity property: if $\infty > T_n$ is a nonincreasing sequence of stopping times converging to $T$, then $\lim_n \mathbb{E}[Y_{T_n}] = \mathbb{E}[Y_T]$. Consequently $Y_{T_n}$ tends to $Y_T$ in $\mathbf{L}^1$.*

*Proof.* Because of the corollary 5.3, we have for each $\epsilon > 0$ the existence of a stopping time $\sigma \ge T$ and $Z \in \mathcal{S}$, so that $\mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\right] \le \mathbb{E}[Y_T] + \epsilon$. Because of the right continuity of the process $X$ we can also suppose that $\sigma > T$. Indeed on the set $\{\sigma = T\}$ we can replace $\sigma$ by $\sigma + \delta$ where $\delta$ is small enough. This not only uses the right continuity of the process $X$, it also uses that for $\delta \to 0$, we have that $\frac{Z_{T+\delta}}{Z_T} \to 1$ in $\mathbf{L}^1$. So we may and do suppose that $\sigma > T$. The following sequence of inequalities is now clear

$$\begin{aligned}
\epsilon + \mathbb{E}[Y_T] &\ge \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\right] \\
&= \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma \ge T_n}\right] + \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma < T_n}\right] \\
&= \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_{T_n}}\frac{Z_{T_n}}{Z_T}\mathbf{1}_{\sigma \ge T_n}\right] + \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma < T_n}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_{T_n}}\mathbf{1}_{\sigma \ge T_n} \mid \mathcal{F}_{T_n}\right]\frac{Z_{T_n}}{Z_T}\right] + \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma < T_n}\right] \\
&\ge \mathbb{E}\left[Y_{T_n}\mathbf{1}_{\sigma \ge T_n}\frac{Z_{T_n}}{Z_T}\right] + \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma < T_n}\right] \\
&\ge \mathbb{E}\left[Y_{T_n}\right] + \ge \mathbb{E}\left[Y_{T_n}\left(\mathbf{1}_{\sigma \ge T_n}\frac{Z_{T_n}}{Z_T} - 1\right)\right] + \mathbb{E}\left[X_\sigma \frac{Z_\sigma}{Z_T}\mathbf{1}_{\sigma < T_n}\right].
\end{aligned}$$

Now the second term tends to zero as $n \to \infty$. Indeed $\frac{Z_{T_n}}{Z_T} \to 1$ in $\mathbf{L}^1$ and the variables $Y_.$ are bounded by the uniform bound on $X$. The third term tends to zero since the set $\{\sigma < T_n\}$ decreases to the empty set. As a result we get that for all $\epsilon > 0$:

$$\epsilon + \mathbb{E}[Y_T] \ge \lim_n \mathbb{E}[Y_{T_n}].$$

The last statement is easy since the sequence $Y_T, (Y_{T_n})_n$ forms a submartingale that has a right continuous modification (see [DM]). This completes the proof of the lemma. □

**Lemma 5.8.** *There is a càdlàg process $V$ so that for all stopping times $T < \infty$ we have that $V_T = Y_T$ a.s. .*

*Proof.* This follows from the modification theorem for submartingales. For the appropriate version (stated for supermartingales) see [DM] page 73, Théorème 1. We first take variables $Y_t$ for each rational $t$, then we apply the modification theorem.

This gives us a càdlàg $\mathbb{P}-$submartingale process $V$ so that for all $t \in \mathbb{R}_+$: $V_t = Y_t$. We still have to prove that $V_T = Y_T$ for finite stopping times. This is not difficult. The equality for deterministic times $t$ implies the equality for stopping times taking rational values. Now take a sequence of stopping times $T_n$ decreasing to $T$ and so that each $T_n$ is finite and takes only rational values. We then have

$$V_T = \lim V_{T_n} = \lim Y_{T_n} = Y_T,$$

where the limits are taken in $\mathbf{L}^1$ and where the last equality follows from the previous lemma.

The proof of the theorem is now complete. It is sufficient to take the process V constructed above as a version for the "process" $\Psi(X)$ and to apply the lemma 5.6.

*Remark.* The notation $\Psi$ is reserved for processes, whereas the notation $\Phi$ was reserved for random variables. The construction is also different in the sense that in the construction of $\Psi$, we use an infinum over stopping times as well as an infimum over all elements of $\mathcal{S}$. If with a random variable $f \in \mathbf{L}^\infty$, we associate the càdlàg process $X_t = \mathbb{E}_\mathbb{P}[f \mid \mathcal{F}_t]$, we could associate with any random variable a process $\Psi_t(f)$ (defined as $\Psi(f) = \Psi(X)$). In the next section, we will give conditions under which, for random variables, both families $\Phi_T(f)$ and $\Psi_T(X)$ are the same. For the moment let us show the following property

**Lemma 5.9.** *Let $\mathcal{S}$ be m–stable. With the notation introduced above, i.e. $X_t = \mathbb{E}_\mathbb{P}[f \mid \mathcal{F}_t]$, we have for every bounded random variable $f$ that*

$$\Psi_t(X) = \Phi_t(f).$$

*Proof.* We obviously have that

$$\Psi_\sigma(X) = \text{ess.inf}_{\sigma \le \tau, \mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{P}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \le \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\sigma] = \Phi_\sigma(f).$$

Indeed take $\tau = \infty$ as a special case in the left hand side.
Conversely, we have that

$$\begin{aligned}
\Psi_\sigma(X) &= \text{ess.inf}_{\sigma \le \tau, \mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{P}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&\ge \text{ess.inf}_{\tau \ge \sigma} \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[\text{ess.inf}_{\mathbb{Q}' \in \mathcal{S}} \mathbb{E}_{\mathbb{Q}'}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&\ge \text{ess.inf}_{\tau \ge \sigma} \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&\ge \text{ess.inf}_{\tau \ge \sigma} \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[f \mid \mathcal{F}_\sigma\right] \\
&\ge \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}\left[f \mid \mathcal{F}_\sigma\right] = \Phi_\sigma(\Phi_\tau(f)). \quad \square
\end{aligned}$$

*Examples.* If $\mathcal{S} = \{\mathbb{P}\}$ then the process $\Psi(X)$ coincides with the Snell envelope (up to minus sign, since we take the lower envelope and not the upper envelope. As well known this "upside down" Snell envelope can be calculated as $\Psi_\sigma(X) = \text{ess.inf}_{\sigma<\infty} \mathbb{E}_\mathbb{P}[X_\sigma]$. The other extreme example is when $\mathcal{S}$ is the set of all probability measures, abosukltely continuous wth respect to $\mathbb{P}$. In this case we have

$$\Psi_0(X) = \text{ess.inf}(\inf_{t \ge 0} X_t),$$

the worst possible loss over time and over all "states of nature". This requires some extra proof, since the time where a process attains its minimum (if the minimum is attained) is not a stopping time. (In finance this would have spectacular consequences as we would be able to buy at the lowest price and sell at the highest price, a strategy opening a lot of perspectives). The proof goes as follows. We suppose that the càdlàg process $X$ is bounded by 1. Let $a = \text{ess.inf}(\inf_{t\geq 0} X_t)$. The set, defined for $\varepsilon > 0$:

$$\pi(A) = \{\omega \mid \text{ there is } t \text{ such that } X_t(\omega) < a + \varepsilon\}$$

is the projection of the optional set

$$A = \{(t,\omega) \mid X_t(\omega) < a + \varepsilon\}.$$

It is therefore measurable, i.e in $\mathcal{F}_\infty$. Moreover the stopping time

$$T(\omega) = \inf\{t \mid (t,\omega) \in A\},$$

is well defined and satisfies $\{T < \infty\} = \pi(A)$. Moreover on $\{T < \infty\}$ we have that $X_T \leq a + \varepsilon$, since the process $X$ is càdlàg . We now take $n$ so that $\mathbb{P}[T \leq n] \geq \mathbb{P}[\pi(A)](1-\varepsilon) > 0$. The probability measure $\mathbb{Q}$ is defined as the conditional probability measure with respect to $\{T \leq n\}$, i.e. it has density $Z_\infty = \mathbf{1}_{\{T\leq n\}}/\mathbb{P}[T \leq n]$. As easily seen $\mathbb{E}_\mathbb{Q}[X_{T\wedge n}] \leq a + \varepsilon$. Since $\varepsilon$ was arbitrary we have $\Psi_0(X) = a$. The identification of $\Psi_\tau(X)$ is more or less the same. Roughly speaking we can say that $\Psi_\tau(X)$ is, given the information $\mathcal{F}_\tau$, the infimum of $\{X_t \mid \tau \leq t < \infty\}$. This requires the use of conditional distributions but it can also be defined as

$$\begin{aligned} &\text{ess.inf}\,(\inf_{\tau\leq t<\infty} X_t) = \\ &\qquad \text{ess.sup}\,\{h \mid h \text{ is } \mathcal{F}_\tau \text{ measurable and } h \leq X_t \text{ for all } t \geq \tau\} \end{aligned}$$

This random variable is equal to the random variable $\Psi_\tau(X)$. Since obviously ess.inf $(\inf_{\tau\leq t<\infty} X_t) \leq \Psi_\tau(X)$, it is sufficient to prove that for every stopping time $\infty > \sigma \geq \tau$ we have that $\Psi_\tau(X) \leq X_\sigma$ a.s. . If this would not be true then the set $C = \{\Psi_\tau(X) > X_\sigma\}$ would not be negligible and we could then take the probability measure $\mathbb{Q}$ with density $\mathbf{1}_C/\mathbb{P}[C]$. For this probability we would get

$$\mathbb{E}_\mathbb{Q}[X_\sigma \mid \mathcal{F}_\tau] < \mathbb{E}_\mathbb{Q}[\Psi_\tau(X) \mid \mathcal{F}_\tau] = \Psi_\tau(X),$$

a contradiction to the definition of $\Psi_\tau(X)$.

*Remark.* The reader could ask why we took the infimum over all probabilities in $\mathcal{S}$ and over all stopping times. It can be argued that if the economic agent can choose the stopping time, e.g. to stop a project, it would be more realistic to take the supremum over all stopping times. This gives a mathematical problem that is related to a maximin/minimax strategy. The mathematics involved are more complicated as the outcome is not the result of a concave problem but rather of a concave-convex optimisation problem. This approach clearly makes sense if the economic agent can choose the stopping time. If however the economic agent cannot choose the stopping time or if the stopping time is selected by the "enemy", then the infimum makes more sense. In case the economic agent can choose the stopping time, it becomes interesting to have a look at the following generalisation. For convenience and to facilitate the mathematics, let us suppose that we are working on a finite horizon interval $[0,T]$. The stopping time $\tau \le T$ can be identified with the nondecreasing process $A_t = \mathbf{1}_{\tau \le t}$ and with the property that $A_T = 1$. The convex closed envelope of these processes brings us to the set of all càdlàg , adapted, nondecreasing processes such that $A_T = 1$. The value $A_t$ could describe the amount of the process that is already stopped — closed down — at time $t$. If the set $\mathcal{S}$ is weakly compact in $\mathbf{L}^1$, we can apply the minimax theorem and we get that

$$\inf_{\mathbb{Q}\in\mathcal{S}} \sup_A \mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u \, dA_u\right] = \sup_A \inf_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u \, dA_u\right].$$

We do not develop this theory any further as it does not fit in the approach we present here.

## 6. The Equivalence between the m-stability property, recursivity and time-consistency

The setup of this section is a little bit more general as we will deal with random variables instead of dealing with stochastic processes. Let us first recall some notations. Let $\mathcal{S}$ be a closed convex set of probability measures $\mathcal{S} \subset \mathbf{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. No stability on $\mathcal{S}$ is assumed. If $f$ is a bounded random variable then for each stopping time $T \le \infty$ we denote by $\Phi_T(f)$ the random variable

$$\Phi_T(f) = \text{ess.inf}_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[f \mid \mathcal{F}_T\right].$$

The non linear functional $\Phi_T$ satisfies the following coherence properties. We omit the straightforward proofs.

1. if $f \ge g$ are bounded random variables then $\Phi_T(f) \ge \Phi_T(g)$
2. for $\lambda \ge 0$ and $f \in \mathbf{L}^\infty$, we have that $\Phi_T(\lambda f) = \lambda \Phi_T(f)$, (the same holds for $\lambda$ a nonnegative bounded $\mathcal{F}_T$ measurable random variable)
3. for $f, g$ bounded random variables we have $\Phi_T(f+g) \ge \Phi_T(f) + \Phi_T(g)$
4. if g is a bounded $\mathcal{F}_T$ measurable random variable, then for any bounded random variable $f$ we have $\Phi_T(f+g) = \Phi_T(f) + g$
5. if $f_n$ is a sequence of random variables $1 \ge f_n \ge -1$ then $\Phi_T(\limsup_n f_n) \ge \limsup_n \Phi_T(f_n)$.

**Definition 6.1.** *The set $\mathcal{S}$ is called time consistent if the following holds. For any pair of stopping times $\sigma \le \tau$ and any pair of random variables $f, g \in \mathbf{L}^\infty$, we have that $\Phi_\tau(f) \le \Phi_\tau(g)$ implies that $\Phi_\sigma(f) \le \Phi_\sigma(g)$.*

The following theorem characterises the convex closed sets $\mathcal{S}$ that are also m–stable. This theorem is related to decision theory with multipriors, where the m-stability is referred to as rectangularity. However the technicalities are different from the ones addressed here. See [ES], [Wang]. Especially the last paper is difficult to read since it does not use the structures introduced in general stochastic process theory.

**Theorem 6.2.** *The following are equivalent*

1. *The set $\mathcal{S}$ is m–stable.*

(2) *For every bounded random variable $f$, the family $\Phi_T(f)$ satisfies: for every two stopping times $\sigma \le \tau$ we have $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f))$.*
(3) *For every bounded random variable $f$, for every stopping time $\sigma$ we have $\Phi_0(f) \le \Phi_0(\Phi_\sigma(f))$.*
(4) *The set $\mathcal{S}$ is time consistent.*
(5) *The family $\Phi_T(f)$ satisfies the submartingale property: for all $\mathbb{Q} \in \mathcal{S}$ and all pair of stopping times $\sigma \le \tau$ we have that $\Phi_\sigma(f) \le \mathbb{E}_\mathbb{Q}[\Phi_\tau(f) \mid \mathcal{F}_\sigma]$.*

*Proof.* First of all let us show that 1) implies 2). So let $\sigma \le \tau$ be two stopping times and let $f$ be a bounded random variable. By lemma 5.4 and the lattice property of lemma 5.2, we have that

$$\begin{aligned}
\text{ess.inf}_\mathbb{Q} \mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\sigma] &= \text{ess.inf}_\mathbb{Q} \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&= \text{ess.inf}_\mathbb{Q} \, \text{ess.inf}_{\mathbb{Q}^1} \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_{\mathbb{Q}^1}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&= \text{ess.inf}_\mathbb{Q} \mathbb{E}_\mathbb{Q}\left[\text{ess.inf}_{\mathbb{Q}^1} \mathbb{E}_{\mathbb{Q}^1}[f \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma\right] \\
&= \text{ess.inf}_\mathbb{Q} \mathbb{E}_\mathbb{Q}\left[\Phi_\tau(f) \mid \mathcal{F}_\sigma\right] \\
&= \Phi_\sigma(f).
\end{aligned}$$

Let us now show the equivalence between 2 and 4. Suppose that for two bounded random variables $f, g$ and two stopping times $\sigma \le \tau$, we have $\Phi_\tau(f) \le \Phi_\tau(g)$. Then we have, because of 4, $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f)) \le \Phi_\sigma(\Phi_\tau(g)) = \Phi_\sigma(f)$. Conversely to prove 4 out of 2, we take for the two random variables, the functions $f$ and $g = \Phi_\tau(f)$. We have equality $\Phi_\tau(\Phi_\tau(f)) = \Phi_\tau(f)$ and therefore (applying item 2 twice) that $\Phi_\sigma(\Phi_\tau(f)) = \Phi_\sigma(f)$.
Obviously we have that 2 implies 3.
We now come to the proof that 3 implies 1, this is the most serious part of the proof. So let us suppose that $Z^1$ and $Z^2$ are two elements in $\mathcal{S}$ – coming from the measures $\mathbb{Q}^1, \mathbb{Q}^2$ – and let $\sigma$ be a stopping time. Also suppose that the element $Z^1_\sigma \frac{Z^2_\infty}{Z^2_\sigma}$ is not in the closed convex set $\mathcal{S}$. By the Hahn–Banach theorem, there is a random variable $f \in \mathbf{L}^\infty$, so that

$$\mathbb{E}_\mathbb{P}\left[Z^1_\sigma \frac{Z^2_\infty}{Z^2_\sigma} f\right] < \inf_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_\mathbb{Q}[f].$$

We can write the left hand side as $\mathbb{E}_{\mathbb{Q}^1}\left[\mathbb{E}_{\mathbb{Q}^2}[f \mid \mathcal{F}_\sigma]\right]$. This is clearly at least equal to $\mathbb{E}_{\mathbb{Q}^1}[\Phi_\sigma(f)]$, a quantity at least equal to $\Phi_0[\Phi_\sigma(f)]$, hence by property 3, at least equal to $\Phi_0(f)$. This is a contradiction since the right hand side is precisely $\Phi_0(f)$.
We still have to show the equivalence with property 5. That 1 implies 5 follows from Theorem 5.1 (see lemma 5.6) of the previous section and lemma 5.9. Suppose now that 5 holds. We have to show property 2. This means for every $f \in \mathbf{L}^\infty$ and every stopping time $\sigma$, we have the inequality $\Phi_0(f) \le \Phi_0(\Phi_\sigma(f))$. Now this inequality follows from the submartingale property for the family $\Phi_T(f)$, $T$ a stopping time. Indeed, for every $\mathbb{Q} \in \mathcal{S}$ we have, by the submartingale property, that $\Phi_0(f) \le \mathbb{E}_\mathbb{Q}[\Phi_\sigma(f)]$. Taking the infimum over all elements $\mathbb{Q} \in \mathcal{S}$ then gives $\Phi_0(f) \le \Phi_0(\Phi_\sigma(f))$, as desired. This completes the proof. □

Again, we can give a version of the above theorem when risk adjusted processes are used. Before we give the precise statement, we need the following proposition.

**Proposition 6.3.** *Let $\mathcal{S}$ be a convex closed set of probability measures, all of them absolutely continuous with respect to $\mathbb{P}$, i.e. $\mathcal{S} \subset \mathbf{L}^1$. Let $X$ be a bounded optional, càdlàg process. Then there is a càdlàg process $Y \le X$ that is a $\mathbb{Q}-$submartingale for each $\mathbb{Q} \in \mathcal{S}$ and such that every other càdlàg process $V \le X$, that is a $\mathbb{Q}-$submartingale for each $\mathbb{Q} \in \mathcal{S}$, is necessarily smaller than $Y$, i.e. satisfies $V \le Y$.*

*Proof.* The proof is a standard construction, we only give a sketch. We look at the set

$$\mathcal{V} = \{V \mid V \text{ is a càdlàg submartingale for each } \mathbb{Q} \in \mathcal{S}, V \le X\}.$$

This set is nonempty since the process $X$ is bounded from below by a constant. If $V^1$ and $V^2$ are both elements in $\mathcal{V}$, then $V^1 \vee V^2$ is still in $\mathcal{V}$. It follows that there is a sequence $V^n \in \mathcal{V}$ such that for each rational $t$ we have that $\mathbb{E}_{\mathbb{P}}[V_t^n]$ tends to $\sup\{\mathbb{E}_{\mathbb{P}}[V_t] \mid V \in \mathcal{V}\}$. Let $Y'_t = \sup_n V_t^n$. The family $(Y'_t)_{t \text{ rational}}$ is still a $\mathbb{Q}-$ submartingale for each $\mathbb{Q} \in \mathcal{S}$. For every $t$ we now define the a.s. limit $Y_t = \lim_{s \downarrow t, s>t} Y'_s$. The process $Y$ is right continuous and is a $\mathbb{Q}$ submartingale for each $\mathbb{Q} \in \mathcal{S}$. It is therefore càdlàg . The process $Y$ is smaller than the process $X$ and every càdlàg process $V \in \mathcal{V}$ necessarily satisfies $V \le Y$. □

*Remark.* In the same way as in the proposition, we can construct a smallest processs, that is bigger than $X$ and that is a $\mathbb{Q}-$supermartingale for each $\mathbb{Q} \in \mathcal{S}$. In particular we can make the following construction. For $f \in \mathbf{L}^\infty$ we define $M$ to be the càdlàg version of the $\mathbb{P}-$martingale $M_t = \mathbb{E}_{\mathbb{P}}[f \mid \mathcal{F}_t]$. Then we construct a process càdlàg $F$ such that for each $\mathbb{Q} \in \mathcal{S}$, the process $F$ is a $\mathbb{Q}-$supermartingale, $F \ge M$ and $F$ is minimal for these properties. The reader can check that the construction yields that $F_\infty = f$, but it might happen that the process $F$ has a jump at time $\infty$. The reader can also check that $F$ is the smallest process such that $F$ is a $\mathbb{Q}-$supermartingale for each $\mathbb{Q} \in \mathcal{S}$ and $F_\infty \ge f$. We will make use of this construction in the next theorem.

**Theorem 6.4.** *The following are equivalent*

(1) *The set $\mathcal{S}$ is m–stable*
(2) *The family of random variables defined for stopping times $T$, as*

$$\Psi_T(X) = \text{ess.inf}\,\{\mathbb{E}_{\mathbb{Q}}[X_\tau \mid \mathcal{F}_T] \mid \tau \ge T \textit{ is a stopping time and } \mathbb{Q} \in \mathcal{S}\}$$

*satisfies $\Psi_T(X) = Y_T$, where $Y$ is the process introduced in the previous proposition, i.e. $Y$ is the biggest càdlàg process, smaller than $X$, that is a $\mathbb{Q}-$submartingale for each $\mathbb{Q} \in \mathcal{S}$.*

*Proof.* We first prove that 1 implies 2. Since by construction we always have that $\Psi_T(X) \ge Y_T$, it suffices to observe that by theorem 5.1, $\Psi(X)$ defines a càdlàg process that is a $\mathbb{Q}-$ submartingale for each $\mathbb{Q} \in \mathcal{S}$.
The converse implication goes as follows. We take a bounded random variable $f$ and define $F$ as in the preceding remark. This means that $F$ is a $\mathbb{Q}-$supermartingale for each $\mathbb{Q} \in \mathcal{S}$, $F_\infty \ge f$ and $F$ is minimal for these properties. We now construct the process $\Psi(F)$ and observe that by hypothesis, $\Psi(F)$ is a $\mathbb{Q}-$submartingale for each $\mathbb{Q} \in \mathcal{S}$. However we have that for each stopping time $\sigma$:

$$\Psi_\sigma(F) = \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}; \tau \ge \sigma}\, \mathbb{E}_{\mathbb{Q}}[F_\tau \mid \mathcal{F}_\sigma] = \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}}\, \mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_\sigma] = \Phi_\sigma(f),$$

where $\tau$ is stopping time and where the second inequality follows from the fact that for each $\mathbb{Q} \in \mathcal{S}$, the process $F$ is a $\mathbb{Q}-$supermartingale. This means that the family $\Phi_\sigma(f)$, where $\sigma$ runs through the set of stopping times, satisfies the submartingale inequality and hence by theorem 6.2, the set $\mathcal{S}$ is m–stable. $\square$

## 7. The Chractaerisatiion using the cone of Acceptable Elements.

From [Delb] we recall that there is a one to one correspondence between risk measures and weak* closed cones of $\mathbf{L}^\infty$. The question arises how we can characterise m–stable sets using the cone of acceptable elements. This question was addressed in [ADEHK xxx] for the discrete time case. The proofs can be copied without big changes. Let us start with some definition and notation.

**Definition 7.1.** *If $\mathcal{S} \subset \mathbf{L}^1(\mathbb{P})$ is a closed, convex set of probability measures, $\mathbb{P} \in \mathcal{S}$, then with $\mathcal{A}$ we denote the set of acceptable elements:*

$$\mathcal{A} = \{f \mid f \in \mathbf{L}^\infty, \text{ for all } \mathbb{Q} \in \mathcal{S} : \mathbb{E}_\mathbb{Q}[f] \geq 0\}.$$

*Of course, in the case where $\mathcal{F}_0$ is trivial, we could also have required that $\Phi_T(f) \geq 0$.*

AS obeserved in [Delb] we have that $\mathcal{A}$ is weak* closed in $\mathbf{L}^\infty$ and we can recover $\mathcal{S}$ as

$$\mathcal{S} = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}\text{: } \mathbb{Q} \text{ is a probability measure and for all } f \in \mathcal{A} : \mathbb{E}_\mathbb{Q}[f] \geq 0\}.$$

**Definition and Notation 7.2.** *Let $\tau$ be a stopping time. An element $f \in \mathbf{L}^\infty(\mathcal{F}_\tau) \cap \mathcal{A} = \mathcal{A}_\tau$ is called $\tau-$acceptable. The set $\mathcal{A}'_\tau$ is defined as*

$$\mathcal{A}'_\tau = \left\{f + g \mid f \in \mathcal{A}_\tau; g \in \mathbf{L}^\infty_+(\mathcal{F}_T)\right\}.$$

**Definition 7.3.** *Let $\tau$ be a stopping time. An element $f \in \mathbf{L}^\infty(\mathcal{F}_T)$ is called acceptable at time $\tau$ if for every event $A \in \mathcal{F}_\tau$ we have that $f\mathbf{1}_A \in \mathcal{A}$. By $\mathcal{A}^\tau$ we denote the set of all elements that are acceptable at time $\tau$:*

$$\mathcal{A}^\tau = \{f \mid \text{ for all } A \in \mathcal{F}_\tau; f\mathbf{1}_\tau \in \mathcal{A}\}.$$

The interpretation of both definitions is straightforward. An element $f$ is acceptable at time $\tau$ if given the information at time $\tau$, the element $f$ is still acceptable. In could happen that an element $f$ is acceptable at time 0, i.e. $f \in \mathcal{A}$, but as uncertainty is revealed and $A \in \mathcal{F}_\tau$ is realised, we see that the "bad" part of $f$ is realised and hence at time $\tau$ the random variable $f$ or better $f\mathbf{1}_A$, should be considered as unacceptable. The following characterisation is straightforward

**Proposition 7.4.** *Let $\tau$ be a stopping time. We have that*

$$\mathcal{A}^\tau = \{f \mid \Phi_\tau(f) \geq 0\}.$$

*Proof.* If $\Phi_\tau(f) \geq 0$, then we have for all $\mathbb{Q} \in \mathcal{S}$ and all event $A \in \mathcal{F}_\tau$ that $\mathbb{E}_\mathbb{Q}[\Phi_\tau(f)\mathbf{1}_A] \geq 0$. By the definition of $\Phi$ this implies that for all $\mathbb{Q} \in \mathcal{S}$ and all event $A \in \mathcal{F}_\tau$ we have $\mathbb{E}_\mathbb{Q}[\mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\tau]\mathbf{1}_A]] \geq 0$. This implies that for all $\mathbb{Q} \in \mathcal{S}$ we have $\mathbb{E}_\mathbb{Q}[f\mathbf{1}_A] \geq 0$. Conversely if $f \in \mathcal{A}^\tau$, we must have that for each $\mathbb{Q} \in \mathcal{S}^e$ that $\mathbb{E}_\mathbb{Q}[f\mathbf{1}_A] \geq 0$ for each $A \in \mathcal{F}_\tau$. This implies that $\mathbb{P}$ a.s we have that $\mathbb{E}_\mathbb{Q}[f \mid \mathcal{F}_\tau] \geq 0$. The definition of $\Phi$ then implies that $\Phi_\tau(f) \geq 0$. $\square$

**Corollary 7.5.** *Let $\tau$ be a stopping time. For each $f \in \mathbf{L}^\infty$ we have that $f - \Phi_\tau(f) \in \mathcal{A}^\tau$.*

**Definition 7.6.** *We say that the cone $\mathcal{A}$ of acceptable elements satisfies the decomposition property if for every stopping time $\tau$ we have that $\mathcal{A} \subset \mathcal{A}_\tau + \mathcal{A}^\tau$.*

The interpretation is clear. Every acceptable element can, for every stopping time $\tau$, be decomposed into two elements. The first element is acceptable when the obeservation period is stopped at $\tau$. The second element is acceptable when the observation starts at time $\tau$. Of course, since $\mathcal{A}^\tau, \mathcal{A}_\tau \subset \mathcal{A}$, the definition is equivalent to the statement that $\mathcal{A} = \mathcal{A}^\tau + \mathcal{A}_\tau$ for every stopping time $\tau$. Since trivially $\mathbf{L}^\infty_+(\mathcal{F}_T) \subset \mathcal{A}^\tau$ we have that $\mathcal{A}^\tau + \mathcal{A}_\tau = \mathcal{A}^\tau + \mathcal{A}'_\tau = \mathcal{A}^\tau + \mathcal{A}_\tau + \mathbf{L}^\infty_+(\mathcal{F}_T)$.

**Theorem 7.7.** *Let $\tau$ be a stopping time, then $f \in \mathcal{A}_\tau + \mathcal{A}^\tau$ if and only if $\Phi_\tau(f) \in \mathcal{A}_\tau$.*

*Proof.* One direction of the proof follows from Corollary 7.5. Indeed if $\Phi_\tau(f) \in \mathcal{A}_\tau$, the equality $f = \Phi_\tau(f) + (f - \Phi_\tau(f))$ implies that $f \in \mathcal{A}_\tau + \mathcal{A}^\tau$. The other direction is proved as follows. Let $f = g + h$ where $g \in \mathcal{A}_\tau$ and $h \in \mathcal{A}^\tau$. Because of the superadditivity of the functions $\Phi$ we have $\Phi_\tau(f) \geq \Phi_\tau(g) + \Phi_\tau(h) \geq g$, since $\Phi_\tau(g) = g$ and $\Phi_\tau(h) \geq 0$. Because $g \in \mathcal{A}_\tau$ and because $\Phi_\tau(f)$ is $\mathcal{F}_\tau$ measurable we get $\Phi_\tau(f) \in \mathcal{A}_\tau$. □

**Corollary 7.8.** *The cone $\mathcal{A}_\tau + \mathcal{A}^\tau$ is always weak* closed.*

*Proof.* We apply the criterion of Krein-Smulian (exactly as in [Delb]). So let $f_n \in \mathcal{A}_\tau + \mathcal{A}^\tau$, $\|f_n\|_\infty \leq 1$, be a sequence of functions, tending a.s to a function $f$. Since $\limsup \Phi_\tau(f_n) \leq \Phi_\tau(f)$ we deduce from $\|\Phi_\tau(f_n)\|_\infty \leq \|f\|_\infty \leq 1$ and $\Phi_\tau(f_n) \in \mathcal{A}_\tau$ that also $\Phi_\tau(f) \in \mathcal{A}_\tau$. □

**Theorem 7.9.** *The set $\mathcal{S}$ is m–stable if and only if for each stopping time $\tau$ we have $\mathcal{A} = \mathcal{A}_\tau + \mathcal{A}^\tau$. Or in other words, if and only if $\mathcal{A}$ satisfies the decomposition property.*

*Proof.* Let $\tau$ be a stopping time. We will use the equivalence (1) and (3) of theorem 6.2. If $\mathcal{S}$ is m–stable we have for each $f \in \mathbf{L}^\infty$ that $\Phi_0(\Phi_\tau(f)) = \Phi_0(f)$. Consequently we have that $f \in \mathcal{A}$ implies that $\Phi_\tau(f) \in \mathcal{A}$ and hence in $\mathcal{A}_\tau$. Conversely we deduce from the decomposition property that $\Phi_0(f) \geq 0$ implies that $\Phi_0(\Phi_\tau(f)) \geq 0$. But the translation property then implies that $\Phi_0(f) \leq \Phi_0(\Phi_\tau(f))$ for every $f \in \mathbf{L}^\infty$. □

*Remark.* In [Delb] the theory of general riskmeasures was developed using finitely additive measures instead of using $\sigma-$additive probability measures. It is not clear how to develop a theory of stable sets in this context. The equivalence of the decomposition property with the m–stability gives us an answer. Since the definition of $\mathcal{A}^\tau$ and $\mathcal{A}_\tau$ are purely algebraic, they apply to every cone. So these concepts could be used in Definition 7.6 and Theorem 7.9 to give an alternative definition of m–stability in the case of risk measures that do not necessarily satisfy the Fatou property. We do not pursue this analysis further.

## 8. The Relation with Bellman's Principle.

In this paragraph we prove that the m–stability is equivalent to the validity of Bellman's principle. Especially in the case of Markov processes such a result can be of great importance. In order not to overload the notation in the theorem we suppose that $\mathcal{S}$ is a closed convex set of probability measures, $\mathcal{S} \subset \mathbf{L}^1$ and (as always) $\mathbb{P} \in \mathcal{S}$. We also suppose that $\mathcal{F}_0$ is trivial. For a bounded process $X$ and a stopping time $\sigma$ we defined

$$\Psi_\sigma(X) = \text{ess.inf}_{\mathbb{Q}\in\mathcal{S}^e;\tau\geq\sigma}\, \mathbb{E}_\mathbb{Q}[X_\tau \mid \mathcal{F}_\sigma].$$

We also recall that if $\sigma$ is a stopping time, the process ${}^\sigma X$ is defined as ${}^\sigma X_s = 0$ if $s \leq \sigma$ and ${}^\sigma X_s = X_s - X_\sigma$ if $s \geq \sigma$. The process $X^\sigma$ is defined as $X_s^\sigma = X_s$ if $s \leq \sigma$ and $X_s^\sigma = X_\sigma$ if $s \geq \sigma$. The proof is the same as in [ADEHK2]. It requires the time interval to be closed from the right, i.e. of the form $[0, t]$ where $0 \leq t < +\infty$.

**Theorem 8.1.** *In case the time interval is closed from the right, say* $[0, t]$, *with* $0 \leq t < +\infty$, *the following two properties are equivalent*

(1) $\mathcal{S}$ *is m–stable*
(2) *(Bellman's principle) For every bounded càdlàg adapted process* $X$ *and every finite stopping time* $\tau \leq t$, *we have that*

$$\Psi_0(X) = \Psi_0(X^\tau + \Psi_\tau({}^\tau X)\mathbf{1}_{[\![\tau,t]\!]}).$$

*Proof.* We first show that Bellman's principle implies stability. For $f \in \mathbf{L}^\infty(\mathcal{F}_t)$ we introduce the process $X$ defined as $X_u = \|f\|_\infty$ for $u < t$ and $X_u = f$ for $u \geq t$. The value $\Psi_\tau(X)$ then coincides with the value $\Phi_\tau(f)$ and the Bellman principle gives the recursivity for $\Phi$. According to theorem 6.2 this implies that $\mathcal{S}$ is m–stable.
Conversely let us suppose that $\mathcal{S}$ is m–stable and let us show the Bellman principle. To simplify the notation we will suppose that the measures $\mathbb{Q}, \mathbb{Q}', \mathbb{Q}''$ are taken in $\mathcal{S}^e$, $\sigma \leq \tau \leq t$ are given stopping times and $\nu$ runs through the set of all stopping times $\sigma \leq \nu \leq t$.

$$\begin{aligned}
\Psi_\sigma(X) &= \text{ess.inf}_{\mathbb{Q},\nu\geq\sigma}\, \mathbb{E}_\mathbb{Q}[X_\nu \mid \mathcal{F}_\sigma] \\
&= \text{ess.inf}_{\mathbb{Q},\nu\geq\sigma}\, \mathbb{E}_\mathbb{Q}[\mathbb{E}_\mathbb{Q}[X_\nu \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma] \\
&= \text{ess.inf}_{\mathbb{Q},\nu\geq\sigma}\, \mathbb{E}_\mathbb{Q}[X_\nu \mathbf{1}_{\nu\leq\tau} + \mathbb{E}_\mathbb{Q}[X_\nu \mathbf{1}_{\nu>\tau} \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma] \\
&= \text{ess.inf}_{\mathbb{Q},\nu\geq\sigma}\, \mathbb{E}_\mathbb{Q}[X_\nu \mathbf{1}_{\nu\leq\tau} + \mathbf{1}_{\nu>\tau}(X_\tau + \mathbb{E}_\mathbb{Q}[X_\nu - X_\tau \mid \mathcal{F}_\tau]) \mid \mathcal{F}_\sigma] \\
&= \text{ess.inf}_{\mathbb{Q},\nu\geq\sigma}\, \mathbb{E}_\mathbb{Q}[X_{\nu\wedge\tau} + \mathbf{1}_{\nu>\tau}\mathbb{E}_\mathbb{Q}[X_\nu - X_\tau \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma]
\end{aligned}$$

The Lemma 5.4 allows us to rewrite the result of the simple ess.inf as a compounded expression:

$$\begin{aligned}
\Psi_\sigma(X) = \text{ess.inf}_{\mathbb{Q}',\nu\geq\sigma}\, &\mathbb{E}_{\mathbb{Q}'}[X_{\nu\wedge\tau} \\
&+ \mathbf{1}_{\nu>\tau}\, \text{ess.inf}_{\mathbb{Q}'',\nu'\geq\tau}\, \mathbf{1}\nu > \tau\mathbb{E}_{\mathbb{Q}''}[X_{\nu'} - X_\tau \mid \mathcal{F}_\tau]) \mid \mathcal{F}_\sigma] \\
= \text{ess.inf}_{\mathbb{Q}',\nu\geq\sigma}\, &\mathbb{E}_{\mathbb{Q}'}[X_\nu \mathbf{1}_{\nu<\tau} + \mathbf{1}_{\nu\geq\tau}(X_\tau + \Psi_\tau({}^\tau X) \mid \mathcal{F}_\sigma] \\
= \text{ess.inf}_{\mathbb{Q}',\nu\geq\sigma}\, &\mathbb{E}_{\mathbb{Q}'}[(X^\tau + \Psi_\tau({}^\tau X))_\nu \mid \mathcal{F}_\sigma] \\
= \Psi_\sigma(X^\tau &+ \Psi_\tau({}^\tau X))
\end{aligned}$$

□

*Remark and Counter-example.* An analysis of the proof shows that if the time interval is not closed on the right, the Bellman principle still follows from the fact that the set $\mathcal{S}$ is stable. Whether the Bellman principle implies the stability property is a much more delicate problem. We will give two answers. In case the set $\mathcal{S}$ is weakly compact in $\mathbf{L}^1$, the answer is yes. Afterwards we will give a counter-example in the case where $\mathcal{S}$ is not weakly compact.

**Proposition 8.2.** *Suppose that the time interval is $\mathbb{R}_+$, suppose that the Bellman principle holds and suppose that the set $\mathcal{S}$ is weakly compact in $\mathbf{L}^1$, then the set $\mathcal{S}$ is m–stable.*

*Proof.* We will adapt the proof of theorem 8.1 above. The idea is to show that $\Phi_0(f) = \Phi_0(\Phi_\tau(f))$ for every finite stopping time $\tau$ and for every bounded function $f$ that is $\mathcal{F}_\infty-$measurable. Since $\mathcal{S}$ is weakly compact thet set

$$\{Z_\sigma \mid \sigma \text{ a finite stopping time, } Z \in \mathcal{S}\}$$

is still relatively weakly compact. If we replace $f$ by the sequence $f_n = \mathbb{E}_\mathbb{P}[f \mid \mathcal{F}_n]$ then *uniformly* for $\mathbb{Q} \in \mathcal{S}$, $f_n$ approximates $f$ in $\mathbf{L}^1(\mathbb{Q})$. It follows that $\Phi_0(f_n), \Phi_\tau(f_n), \Phi_0(\Phi_\tau(f_n))$ tend to $\Phi_0(f), \Phi_\tau(f), \Phi_0(\Phi_\tau(f))$. It is therefore sufficient to prove the statement for functions that are $\mathcal{F}_n-$measurable. This is done exactly in the same way as in the proof of the theorem. □

It is clear that a counter-example will have to use the fact that the set $\mathcal{S}$ is big. For notational ease we will work on the time interval $[0, 1[$. This is equivalent to the time interval $\mathbb{R}_+$, just use a time transform $u = t/(t + 1)$. The use of the time interval $[0, 1[$ allows us to use a Brownian Motion $W$ defined for all times $t < \infty$ even if we only need the part before time 1. Finite stopping times will now be replaced by stopping times $\nu < 1$. The filtration we will use is the usual filtration coming from the process $W$. The set $\mathcal{S}$, we will use, is defined as

$$\{Z_1 \mid \mathbb{E}_\mathbb{P}[Z_1] = 1, Z_1 \geq 0, \mathbb{E}_\mathbb{P}[Z_1 \operatorname{sign}(W_1)] = 0\}.$$

It is clear that this set is not m–stable. This can be seen using the definition of m–stability but it will follow from the results below. We first give the sequence of lemma's used to prove the Bellman principle and then we will give the details of the proofs of these lemma's. Since the Bellman principle will be valid and will in fact be equivalent to the risk adjusted value

$$\Psi_0(X) = \text{ess.inf}\{\inf_{0 \leq t < 1} X_t\},$$

we cannot have m-stability. Indeed $\Phi_0(f) = 0$ for $f = \operatorname{sign}(W_1)$.

**Lemma 8.3.** *The $\nu < 1$ be a stopping time, then the set*

$$\{Z_\nu \mid Z \in \mathcal{S}\}$$

*is dense in the set of all $\mathcal{F}_\nu$ measurable densities of probabilities absolutely continuous with respect to $\mathbb{P}$.*

**Lemma 8.4.** *Bellman's principle is valid.*

**Lemma 8.5.** *Let $\mathcal{Q}$ be the set of all density processes $Z$ such that*

(1) $Z_1 = \mathcal{E}(q \cdot W)_1 > 0$, $\mathbb{E}_\mathbb{P}[Z_1] = 1$
(2) $\int_0^1 q_u \, du = 0$ *a.s.* .

*Then we have that $\mathcal{Q} \subset \mathcal{S}$. Let $\tau < 1$ be a stopping time then*

$$\{Z_\tau \mid Z \in \mathcal{Q}\}$$

*is dense in the set of all probability densities on the sigma algebra $\mathcal{F}_\tau$.*

**Lemma 8.6.** *Let $\nu < 1$ be a stopping time and let $q$ be a predictable process, defined on $[0,1] \times \Omega$ so that*

1. *$q_u = 0$ for $u \le \nu$*
2. *$q$ is measurable for the sigma algebra $\mathcal{R} \times \mathcal{F}_\nu$ where $\mathcal{R}$ is the Borel sigma algebra on $[0,1]$,*
3. *a.s. $\int_\nu^1 q_u^2 \, du < \infty$,*

*then $\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1] = 1$ and therefore $\mathcal{E}(q \cdot W)_1$ is the density of a probability measure, equivalent to $\mathbb{P}$. Moreover we have*

$$\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1 \mid \mathcal{F}_\nu] = 1.$$

*Proof of Lemma 8.6.* This is almost trivial. Seen from time $\nu$ the process $q$ is deterministic. Here are the details. For each $n$ we put

$$A_n = \left\{ \int_\nu^1 q_u^2 \, du \le n \right\}.$$

Clearly $A_n \in \mathcal{F}_\nu$ and the stochastic exponential $\mathcal{E}(\mathbf{1}_{A_n} q \cdot W)$ satisfies Novikov's condition. Therefore we have

$$\mathbb{E}_\mathbb{P}[\mathbf{1}_{A_n} \mathcal{E}(q \cdot W)_1] = \mathbb{E}_\mathbb{P}[\mathbf{1}_{A_n} \mathcal{E}(\mathbf{1}_{A_n} q \cdot W)_1] = \mathbb{P}[A_n].$$

We now apply Beppo Levi's theorem to conclude that $\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1] = 1$ as desired. The statement on the conditional expectation follows from the fact that since $\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1] = 1$, $\mathcal{E}(q \cdot W)$ must be a uniformly integrable martingale. □

*Proof of Lemma 8.5 and 8.3.* Let $Z_\tau$ be the density of a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_\tau$. The process $Z$ is supposed to be defined up to time $\tau$. We will now extend it in such a way that it defines an element $Z \in \mathcal{Q}$. The process $Z$ is a stochastic exponential and therefore $Z_\tau$ can be written as $Z_\tau = \mathcal{E}(q \cdot W)_\tau$. The predictable process $q$ is defined up to time $\tau$. Since $Z_\tau > 0$ we must have that $\int_0^\tau q_u^2 \, du < \infty$ and therefore we also have that $r = \int_0^\tau q_u \, du$ is defined. If we now put for $u > \tau$

$$q_u = \frac{-r}{1-\tau}$$

we have that $q\mathbf{1}_{]\tau,1[}$ satisfies the assumptions of lemma 8.6. We therefore have that

$$\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1] = \mathbb{E}_\mathbb{P}[\mathbb{E}_\mathbb{P}[\mathcal{E}(q \cdot W)_1 \mid \mathcal{F}_\tau]] = \mathbb{E}_\mathbb{P}[1] = 1.$$

Moreover $\int_0^1 q_u \, du = \int_0^\tau q_u \, du + \int_\tau^1 q_u \, du = r + (-r) = 0$. Also we have that $\int_0^1 q_u^2 \, du = \int_0^\tau q_u^2 \, du + \int_\tau^1 q_u^2 \, du = \int_0^\tau q_u^2 \, du + r^2/(1-\tau) < \infty$. Therefore $Z_1 > 0$ and $Z \in \mathcal{Q}$. This proves the density part of the lemma. We now prove that $\mathcal{Q} \subset \mathcal{S}$. For an element $\mathbb{Q} \in \mathcal{Q}$ we have that $W$ is a Brownian motion with drift $q_u \, du$. Therefore the variable $W_t$ is, under the measure $\mathbb{Q}$, equal to a gaussian random variable + $\int_0^t q_u \, du$. For $t = 1$ this simply means that under $\mathbb{Q}$, the random variable $W_1$ is still a symmetric gaussian random variable with $\mathbf{L}^2(\mathbb{Q})$ norm 1. In particular we have that $\mathbb{E}_\mathbb{Q}[\text{sign}(W_1)] = 0$, i.e. $\mathbb{Q} \in \mathcal{S}$. Lemma 8.3 immediately follows from Lemma 8.5. □

*Proof of lemma 8.4.* Let us suppose that $X$ is càdlàg , bounded adapted. Furthermore let us fix a stopping time $\nu < 1$. It is clear that

$$\Psi_\nu(X) = X_\nu + \Psi_\nu({}^\nu X).$$

So we have to calculate $\Psi_\nu({}^\nu X)$. By definition we have

$$\Psi_\nu({}^\nu X) = \text{ess.inf}_{\nu \le \sigma < 1} \, \text{ess.inf}_{\mathbb{Q} \in \mathcal{S}^e} \left\{ \mathbb{E}_\mathbb{Q}[{}^\nu X_\sigma \mid \mathcal{F}_\nu] \right\}.$$

Because of lemma 8.3 this can also be written as

$$\Psi_\nu({}^\nu X) = \text{ess.inf}_{\nu \le \sigma < 1} \, \text{ess.inf}_{\mathbb{Q} \sim \mathbb{P}} \left\{ \mathbb{E}_\mathbb{Q}[{}^\nu X_\sigma \mid \mathcal{F}_\nu] \right\}.$$

Indeed the set

$$\{Z_\sigma \mid Z \in \mathcal{S}^e\}$$

is dense in the set

$$\{Z_\sigma \mid Z \text{ a nonnegative uniformly integrable martingale with } \mathbb{E}_\mathbb{P}[Z_1] = 1\}.$$

This means that the $\Psi-$operator is the same when calculated with the set $\mathcal{S}$ as with the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$. The latter set is stable and therefore the $\Psi-$ operator satisfies Bellman's inequality. □

We end these analysis with the following

**Corollary 8.7.** *The m–stable hull of the set $\mathcal{Q}$ is the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$.*

## 9. The Set of Local Martingale Measures for a finite dimensional locally bounded Price Process.

In this section we will prove that for locally bounded processes, the set of martingale measures forms an m–stable set. This allows us to apply our previous results to situations occurring in finance. We will also see what m–stable sets can occur as sets of martingale measures for finite dimensional processes. The latter characterisation is not fully complete since it will only be done in the context of continuous filtrations. Throughout this section we will use the following notation, see [DS] for more information.

On the filtered probability space $\left(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}\right)$, let $S: \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be an adapted càdlàg process that takes values in the $d-$dimensional space $\mathbb{R}^d$. We suppose that the process is locally bounded and that the original measure is a local martingale measure for the process $S$. This is a simplification when compared to the situation in finance, but it simplifies notation without destroying its generality. Since the process $S$ is locally bounded, the set

$$\mathcal{S} = \{\mathbb{Q} \ll \mathbb{P} \mid \text{ the process } S \text{ is a local martingale for } \mathbb{Q}\}$$

is a closed convex set. As the following shows, it is also m–stable..

**Proposition 9.1.** *The set $\mathcal{S}$ is m–stable.*

*Proof.* We can suppose that the process $S$ is bounded (in the same way as in [DS]). That the set $S$ is convex and closed is then obvious. The m–stability is also quite obvious. Let us take $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{S}^e$. Let $Z^1, Z^2$ be the associated density processes. If $\sigma$ is a stopping time, we have to show that the density process defined as $Z_t = Z^1_t$ for $t \geq \sigma$ and $Z_t = Z^1_\sigma \frac{Z^2_t}{Z^2_\sigma}$, is still in $\mathcal{S}$. To show this, it is sufficient to show that the process $ZS$ is a $\mathbb{P}-$martingale. This is easy. Indeed first observe that the process $Z^1 S$ is a $\mathbb{P}-$martingale (since $\mathbb{Q}^1 \in \mathcal{S}$). The same applies to $Z^2$ and hence the process $\mathbf{1}_{t\geq\sigma}\left(Z^2_t S_t - Z^2_\sigma S_\sigma\right)$ is also a $\mathbb{P}-$martingale. It follows that the process:

$$Z_t S_t = Z^1_{t\wedge\sigma} S_{t\wedge\sigma} + \frac{Z^1_\sigma}{Z^2_\sigma}\mathbf{1}_{t\geq\sigma}\left(Z^2_t S_t - Z^2_\sigma S_\sigma\right)$$

is also a $\mathbb{P}-$martingale. □

To avoid complicated notation we first introduce some extra notions. We restrict ourselves to the case of a continuous price process $S$. As above we may and do suppose that $S$ is bounded. If $X$ is a local martingale then there is a decomposition of $X$ with respect to $S$. This decomposition, called the Kunita-Watanabe-Galtchouk decomposition, allows to write $X$ as a sum of two local martingales. One is a stochastic integral with respect to $S$, the other part $M$ is strongly orthogonal to $S$. So let us write $X = H \cdot S + M$. Saying that $X$ is strongly orthogonal to $S$ means that $H \cdot S$ is strongly orthogonal to $S$. This means that the vector $H$ is orthogonal to the predictable range of $S$. In other words it means that the measure $H'\, d\langle S, S\rangle H = 0$ and this implies that $H \cdot S = 0$. This can only happen when the price process has some redundance.

**Theorem 9.2.** *With the notation of the preceding paragraphs and under the assumption that $S$ is continuous we have that*

$$\mathcal{S} = \{\mathcal{E}(X) \mid \mathcal{E}(X)_\infty \geq 0, X \textit{ is strongly orthogonal to } S,\ \mathcal{E}(X) \textit{ is unif. integrable}\}.$$

*Proof.* The proof is very easy. If $\mathcal{E}(X)$ is a uniformly integrable, nonnegative martingale, where $X = H \cdot S + M$ is the Kunita-Watanabe-Galtchouk decomposition, then $\mathcal{E}(X)S$ is a martingale if and only if $X$ is strongly orthogonal to $S$. This is equivalent to $H \cdot S$ being strongly orthogonal to $S$. The latter is equivalent to the fact that every coordinate of $S$ is strongly orthogonal to $H \cdot S$ and hence to the fact that $H'\, d\langle S, S\rangle H = 0$. This in turn is equivalent to the property $PH = H$. □

There is also a converse to this theorem. The interpretation of such a converse theorem is the following. Given a convex closed set of probabilities, when does there exist a finite dimensional process, say $S$, such that the given set is the set of absolutely continuous martingale measures for the process $S$? A necessary condition is certainly that the set is m–stable. In the continuous case the answer is given by the following theorem.

**Theorem 9.3.** *Let $\mathcal{S}$ be a stable set of probability measures. Let the filtration be so that every local martingale is the stochastic integral with respect to the d$-$dimensional local martingale $M$. Let $\mathcal{S}$ be given by the closure of*

$$\mathcal{S}^e = \{\mathcal{E}(q \cdot M)_\infty \ \mid\ q \in \Phi \textit{ and } \mathbb{E}\left[(\mathcal{E}(q \cdot M))_\infty\right] = 1\},$$

*where the set-valued predictable process $\Phi$ is convex and closed valued. Then the set $\mathcal{S}$ is a set of equivalent local martingale measures for a price process if and only if each $\Phi(t,\omega)$ is a subspace. If the predictable projection valued process $P$ is the orthogonal projection on the space $\Phi(t,\omega)$, then the price process $S$ can be chosen as $S = (Id_{\mathbb{R}^d} - P) \cdot M$.*

*Proof.* The proof is a reformulation of the above theorem 8.1 and theorem 3.1. The details are left to the reader. □

*Remark.* the situation can be generalised to the setting of theorem 9.2 above in the sense that we may suppose that $M$ only generates the continuous local martingales. This means that every local martingale is given by a decomposition of the form $H \cdot M + N$, where $N$ is purely discontinuous. In that case we get the following theorem

**Theorem 9.4.** *With the above notation we have that the closure $\mathcal{S}$ of the set*

$$\mathcal{S}^e = \left\{ \mathcal{E}(q \cdot M + N) \,\middle|\, \begin{array}{l} q \in \Phi \\ \mathcal{E}(q \cdot M + N) \text{ uniformly integrable and strictly positive} \\ N \text{ is purely discontinuous} \end{array} \right\},$$

*is a set of risk neutral measures if and only if each $\Phi(t,\omega)$ is a subspace. If the predictable projection valued process $P$ is the orthogonal projection on the space $\Phi(t,\omega)$, then the price process $S$ can be chosen as $S = (Id_{\mathbb{R}^d} - P) \cdot M$.*

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