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Optimal portfolio selection in a Value-at-Risk framework

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Abstract

In this paper, we develop a portfolio selection model which allocates financial assets by maximising expected return subject to the constraint that the expected maximum loss should meet the Value-at-Risk limits set by the risk manager. Similar to the mean–variance approach a performance index like the Sharpe index is constructed. Furthermore when expected returns are assumed to be normally distributed we show that the model provides almost identical results to the mean–variance approach. We provide an empirical analysis using two risky assets: US stocks and bonds. The results highlight the influence of both non-normal characteristics of the expected return distribution and the length of investment time horizon on the optimal portfolio selection. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Modern portfolio theory aims to allocate assets by maximising the expected risk premium per unit of risk. In a mean–variance framework risk is defined in terms of the possible variation of expected portfolio returns. The focus on standard deviation as the appropriate measure for risk implies that investors weigh the probability of negative returns equally against positive returns. However it is a stylised fact that the distribution of many financial return series are non-normal, with skewness and kurtosis pervasive.[1] Furthermore there is ample evidence that agents often treat losses and gains asymmetrically. There is a wealth of experimental evidence for loss aversion (see, for example, Kahneman et al., 1990). The choice therefore of mean–variance efficient portfolios is likely to give rise to an inefficient strategy for optimising expected returns for financial assets whilst minimising risk. It would therefore be more desirable to focus on a measure for risk that is able to incorporate any non-normality in the return distributions of financial assets. Indeed risk measures such as semi-variance were originally constructed in order to measure the negative tail of the distribution separately.

Typically mainstream finance rests on the assumption of normality, so that a move away from the assumption of normally distributed returns is not particularly favoured; one drawback often stated is the loss in the possibility of moving between discrete and continuous time frameworks. However it is precisely this simplifying approach, whereby any deviations from the square root of time rule are ignored, which needs to be incorporated into current finance theory. The ability to focus on additional moments in the return distribution with the possibility of allowing for skewed or leptokurtotic distributions enables additional risk factors (along with the use of standard deviation) to be included into the optimal portfolio selection problem.[2]

In this paper, we develop an optimal portfolio selection model which maximises expected return subject to a downside risk constraint rather than standard deviation alone. In our approach, downside risk is written in terms of portfolio Value-at-Risk (VaR), so that additional risk resulting from any non-normality may be used to estimate the portfolio VaR. This enables a much more generalised framework to be developed, with the distributional assumption most appropriate to the type of financial assets to be employed. We develop a performance index similar to the Sharpe ratio, and for the case that

[1] See among others Fama and Roll (1968), Boothe and Glassman (1987), and Jansen and de Vries (1991).

[2] Recent research by Harvey and Siddique (2000), Bekaert et al. (1998) and Das and Uppal (1999) indeed advocate the need to incorporate non-normalities into the portfolio allocation decision.

financial assets are assumed to be normally distributed, provide a model similar to the mean–variance approach.

The plan of the paper is as follows: We introduce the framework in Section 2. Section 3 then provides empirical results of the optimal portfolio allocation for a US investor. In Section 4 we address the importance of the non-normal characteristics of expected return distributions in such a framework. Conclusions and practical implications are drawn in Section 5.

2. Portfolio selection under shortfall constraints

Portfolio selection under shortfall constraints has its origins in the work by Roy (1952) on safety-first theory. Roy defined the shortfall constraint such that the probability of the portfolio value falling below a specified disaster level is limited to a specified disaster probability. We extend the literature on asset allocation subject to shortfall constraints.[3] We address the potential problem concerning the definition of disaster levels and probabilities through the use of VaR, and develop a market equilibrium model for portfolio selection, which allows for alternative parametric distributions to be used. Banks and financial institutions have adopted VaR as the measure for market risk,[4] whereby VaR is defined as the maximum expected loss on an investment over a specified horizon given some confidence level.[5] For example a 99% VaR for a 10-day holding period,[6] implies that the maximum loss incurred over the next 10 days should only exceed the VaR limit once in every 100 cases. It therefore reflects the potential downside risk faced on investments in terms of nominal losses. Introducing VaR as a shortfall constraint into the portfolio selection decision, so that the portfolio manager or investor is highly concerned about the value of the portfolio falling below the VaR constraint, is much more in fitting with individual perception to risk and more in line with the constraints which management currently face.

In the framework developed, the measure for risk is defined in terms of the VaR over and above the risk free rate of return on the initial wealth. The portfolio is then selected to maximise expected return subject to the level of risk. The final choice of portfolio, including the borrowing and lending deci-

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[3] See also Leibowitz and Kogelman (1991), and Lucas and Klaassen (1998) who, for example, construct portfolios by maximising expected return subject to a shortfall constraint, defined such, that a minimum return should be gained over a given time horizon for a given confidence level.

[4] See Jorion (1997) for a comprehensive introduction into VaR methodology.

[5] In practice these confidence levels for VaR range from 95% through 99%, whereby the Basle Committee recommends 99%.

[6] This is the VaR recommended by the Basle Committee for Banking Regulation used in establishing a bank's capital adequacy requirements.

sion will therefore meet the specified VaR limit. VaR is therefore used as an ex-ante market risk control measure, extending the richness of VaR as a risk management tool. Developing upon the framework as laid out by Arzac and Bawa (1977), we provide a model in terms of downside risk, so that the optimal portfolio is determined in terms of its VaR, and a performance index, similar to the Sharpe ratio is developed. In this way, we are able to leave the distributional assumptions about the structure of the tails of the distribution or any skewness, to that most in accordance with the financial asset held. This has the advantage of allowing for non-normal payoffs as with most derivative products: providing a general but highly desirable model for optimal portfolio selection. We shall also see that under certain distributional assumptions the model collapses to the CAPM,[7] as developed by Sharpe (1964), Lintner (1965) and Mossin (1966). Since the model is able to encompass much of modern portfolio theory we are able to observe the effect on the portfolio decision induced by non-normalities.

3. Portfolio selection model

In this section, we present a portfolio construction model subject to a VaR limit set by the risk manager for a specified horizon. In other words, we derive an optimal portfolio such that the maximum expected loss would not exceed the VaR for a chosen investment horizon at a given confidence level. Using VaR as the measure for risk in this framework is in accordance with the banking regulations in practice and provides a clear interpretation of investors' behaviour of minimising downside risk. The degree of risk aversion is set according to the VaR limit; hence avoiding the limitations of expected utility theory as to the degree of risk aversion, which an investor is thought to exhibit.

3.1. Portfolio selection problem and downside risk constraint

Suppose that we have an amount $W(0)$ to be invested for an investment horizon T, which we want to invest such that the portfolio meets a chosen VaR limit. This could be set for example by the risk management department, so that the financial institution meets the Basle capital adequacy requirement, or by the private investor according to his individual aversion to risk. This amount can therefore be invested along with an amount B representing borrowing ($B > 0$) or lending ($B < 0$). We assume r_f is the interest rate at which the investor can borrow and lend for the period T. There are n available assets,

[7] See Arzac and Bawa (1977) for the derivation of the CAPM.

and $\gamma(i)$ denotes the fraction invested in the risky asset, i. The $\gamma(i)$s must therefore sum to one. Let $P(i,t)$ be the price of asset i at time t (the current decision period is therefore when $t=0$). The initial value of the portfolio, in Eq. (1), represents the budget constraint:

$$W(0) + B = \sum_{i=1}^{n} \gamma(i)P(i,0). \tag{1}$$

The manager or investor therefore needs to choose the fractions $\gamma(i)$ to be invested with the initial wealth $W(0)$ and the amount borrowed or lent at time 0. Allocating the assets in the portfolio and choosing the amount to borrow or lend such that the maximum expected level of final wealth is achieved results in the definition of the portfolio allocation problem. Choosing the desired level of VaR as VaR* we therefore formulate the downside risk constraint as follows:

$$\Pr\{W(0) - W(T,p) \geqslant \mathrm{VaR}^*\} \leqslant (1-c), \tag{2}$$

where Pr denotes the expected probability conditioned on the information available at time zero, for portfolio p. Eq. (2) is equivalent to

$$\Pr\{W(T,p) \leqslant W(0) - \mathrm{VaR}^*\} \leqslant (1-c). \tag{3}$$

Since VaR is the worst loss over the investment horizon T, which can be expected with confidence level c, the investor's level of risk aversion is reflected in both the level of the VaR, and the confidence level associated with it. The optimal portfolio, which is derived such that Eq. (3) holds, will therefore reflect this.

3.2. Optimal portfolio construction

The introduction of VaR however provides us with a shortfall constraint (denoted by Eq. (3)) that fits perfectly into the Arzac and Bawa framework. We therefore build upon their results to derive an optimal portfolio selection model. The investor is interested in maximising wealth at the end of the investment horizon. Let $r(p)$ be the expected total return on a portfolio p in period T; assume that asset i is included with fraction $\gamma(i,p)$ in portfolio p. The expected wealth from investing in portfolio p at the end of the investment horizon becomes

$$E_0(W(T,p)) = (W(0) + B)(1 + r(p)) - B(1 + r_f). \tag{4}$$

Substituting for B and the downside risk constraint (Eq. (3)), the final expected return on wealth is maximised for an investor concerned about the downside risk by the portfolio maximising $S(p)$ in Eq. (5). We denote this maximising portfolio as p', where $q(c,p)$ simply defines the quantile that corresponds to probability $(1-c)$ of occurrence, which can be read off the cdf of the expected return distribution for portfolio p.

$$p' : \max_p S(p) = \frac{r(p) - r_{\mathrm{f}}}{W(0)r_{\mathrm{f}} - W(0)q(c,p)}. \tag{5}$$

Note that although initial wealth is in the denominator of $S(p)$ it does not affect the choice of the optimal portfolio since it is only a scale constant in the maximisation. The asset allocation process is thus independent of wealth. The advantage however of having initial wealth in the denominator is in its interpretation. $S(p)$ equals the ratio of the expected risk premium offered on portfolio p to the risk, reflected by the maximum expected loss on portfolio p that is incurred with probability $1-c$ relative to the risk-free rate. Since the negative quantile of the return distribution multiplied by the initial wealth is the VaR associated with the portfolio for a chosen confidence level, we are able to derive an expression for the risk faced by the investor as φ. Letting VaR (c,p) denote portfolio p's VaR, the denominator of (5) may be written as

$$\varphi(c,p) = W(0)r_{\mathrm{f}} - \mathrm{VaR}(c,p). \tag{6}$$

Such a measure for risk is in fitting with investors' behaviour of focussing on the risk-free rate of return as the benchmark return with risk being measured as the potential for losses to be made with respect to the risk-free rate as the point of reference. Indeed the measure for risk can be seen as a possible measure for regret, since it measures the potential opportunity loss of investing in risky assets. Investors will therefore only accept greater returns if they can tolerate the regret occurring from the greater potential wealth-at-risk. The risk–return ratio $S(p)$, which is maximised for the optimal portfolio p' can therefore be written as

$$p' : \max_p S(p) = \frac{r(p) - r_f}{\varphi(c,p)}. \tag{7}$$

$S(p)$ is thus a performance measure like the Sharpe index that can be used to evaluate the efficiency of portfolios (see Sharpe, 1994, for more details). Indeed under the assumption that expected portfolio returns are normally distributed, and the risk-free rate is zero, $S(p)$ collapses to a multiple of the Sharpe index. In this case the VaR is expressed as a multiple of the standard deviation of the expected returns so that the point at which both performance indices are maximised will lead to the same optimal portfolio being chosen. Only a minimal difference in the optimal portfolio weights occurs for positive risk free rates, for a small time horizon.

The optimal portfolio that maximises $S(p)$ in Eq. (7) is chosen independently from the level of initial wealth. It is also independent from the desired VaR, since the risk measure φ for the various portfolios depends on the estimated portfolio VaR rather than the desired VaR. Investors first allocate the risky assets and then the amount of borrowing or lending will reflect by how much the VaR of the portfolio differs from the VaR limit set; thus two-fund separation holds like in the mean–variance framework. However, since the investors' degree of risk aversion is captured by the chosen VaR level, the amount of borrowing or lending required to meet the VaR constraint may be determined. This is a significant benefit of the model, and therefore has serious practical implications, with investors being able to easily and accurately determine the desired risk–return trade-off with the required amount of borrowing or lending easily determined. The amount to be borrowed is denoted by Eq. (8):

$$B = \frac{W(0)(\mathrm{VaR}^* - \mathrm{VaR}(c,p'))}{\varphi'(c,p')}. \tag{8}$$

The optimal portfolio is independent of the distributional assumption, so that the model has been derived solely on the premise that investors wish to maximise expected return subject to a downside risk constraint.

4. Optimal portfolio selection for US stocks and bonds

In order to determine the effect of deviations from normality, and the time horizon chosen for the VaR level we have estimated the optimal portfolios for a US investor using US Stocks and Bonds such that a VaR constraint over various time horizons is met. We use data obtained from datastream for the S&P 500 composite return index for the US, the 10-year datastream benchmark US government bond return index and the 3-month US Treasury Bill rate for the risk-free rate. We employ daily data from these US indices from January 1990 until December 1998, providing us with 2364 observations. The average annual return on the S&P 500 over the sample period was 16.81%, just over twice as high as the average annual return on the 10-year Government Bond Index of 8.35%. The annual standard deviation is also higher on the S&P 500 at 13.42% per annum, compared to the less volatile nature of the Government Bonds with an annual standard deviation of only 6.31%.

Looking at the alternative frequencies in Table 1, we see that the monthly average return is naturally greater than the daily return; however the standard deviation of the distribution is also greater, and is even greater than the square root of time rule would suggest. This provides an indication of autocorrelation. We also see that for all three data frequencies significant skewness and kurtosis is prevalent.

Table 1
Summary statistics[a]

	Daily	Bi-weekly	Monthly
S&P 500 composite return index			
Observations	2364	248	132
Average return	0.000528	0.00523	0.010804
Standard deviation	0.007717	0.028459	0.037896
Maximum return	0.058101	0.098783	0.106718
Minimum return	−0.03532	−0.07666	−0.11075
Skewness	0.179661	0.170371	−0.06104
Kurtosis	6.78397	3.73567	3.45994
10-year datastream US benchmark Government Bond Index			
Observations	2364	248	132
Average return	0.000331	0.003317	0.006572
Standard deviation	0.003972	0.012421	0.018705
Maximum return	0.016462	0.031485	0.039118
Minimum return	−0.02826	−0.04402	−0.05199
Skewness	−0.39087	−0.38734	−0.46151
Kurtosis	6.23627	3.34004	2.8721

[a] The table gives the summary statistics for the S&P 500 composite returns index and the 10-year datastream US benchmark government bond index over the period January 1990–December 1998.

4.1. Optimal portfolio selection using the empirical distribution

To find the portfolio which maximises the performance index $S(p)$ in (7) we estimate both the expected return $r(p)$ and the VaR for various combinations of US stocks and bonds, using the daily, biweekly and monthly data over the sample period. Plotting the risk–return trade-off provides us with an efficient VaR frontier for a given confidence level for VaR, moving from a portfolio containing 100% bonds to a 100% investment into stocks. In Fig. 1, we have plotted efficient VaR frontiers using daily data, whereby alternative distributional assumptions have been used to estimate φ, the parameter for risk.

The efficient VaR frontier is similar to a mean–variance frontier except for the definition of risk: VaR relative to the benchmark return (φ) instead of standard deviation (σ). The empirical distribution provides the true risk–return trade-off as observed in financial markets; however the greater the time horizon for the investment then the less precise the efficient VaR frontier. In order to determine the exact proportion of the portfolio which needs to be held in cash, we need to know the investor's risk profile. The level set for VaR, which includes the choice for the confidence level associated with the VaR level, determines this. In the empirical example below, we have set the desired VaR level as the 95% VaR from the historical distribution. This provides us with a benchmark with which we can compare the alternative distributional assumptions and various time horizons used for the investment period. An investor who wants to be 95% confident that his or her wealth will not drop by

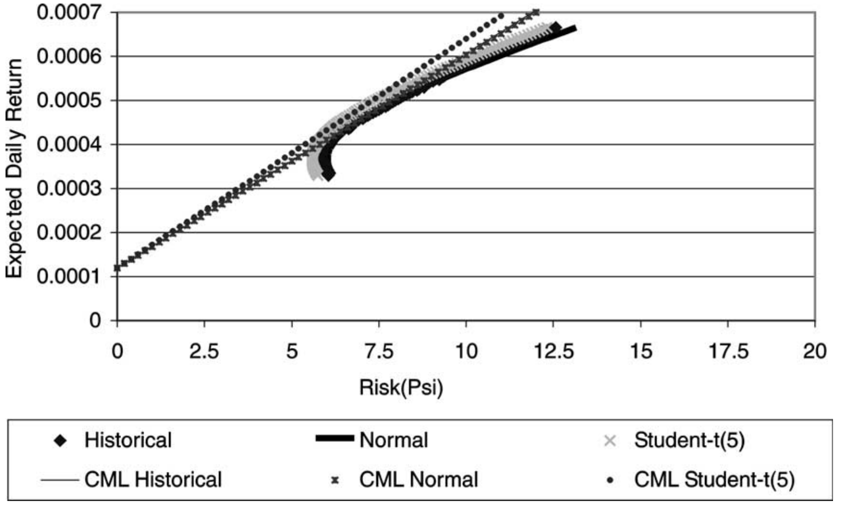

Fig. 1. Efficient VaR frontier – daily data and daily VaR at 95% confidence level.
Note: The figure presents the risk–return trade-off for portfolios of stocks and bonds whereby risk is measured by the downside risk measure φ of the portfolio at the 95% confidence level. The returns and VaR estimates are obtained using daily data on the S&P 500 composite returns index and the 10-year datastream US benchmark government bond index for the period January 1990–December 1998. We present the efficient frontier for the empirical distribution, the parametric normal approach and under the assumption of a student-*t* distribution with 5 degrees of freedom.

more than the daily VaR limit, however attains the highest possible return therefore selects the point on the efficient VaR frontier, where return per unit of risk is maximised. To determine the optimal allocation between stocks and bonds, we set the risk-free rate at 4.47%, the last available 3-month Treasury bill rate in the sample period. For an investor with a VaR limit at the 95% confidence level the optimal allocation between US stocks and bonds occurs when 36% of wealth is held in stocks and 64% in bonds. The combinations for stocks and bonds for a variety of confidence levels are provided in the first two columns of Table 2, and the portfolio VaR is given in the third column.

Naturally the greater the confidence level chosen in association with the VaR then the greater the portfolio VaR. Absolute portfolio VaR is given in the final column. In order to ensure that the portfolio meets the desired VaR, in accordance with Eq. (8) a greater proportion of the portfolio will be needed to be held at the risk free rate the higher the confidence level associated with the VaR level set: a movement along the Capital Market Line, also shown in Fig. 1. The final proportions of the portfolio optimising the risk–return trade-off for the chosen VaR level are provided in Table 3. We can see how sensitive the portfolio selection decision is to changes in the confidence level associated with the VaR limit.

Table 2
Optimal portfolios using empirical distribution for VaR estimation[a]

Confidence level (%)	Stocks (%)	Bonds (%)	Portfolio VaR ($)
Daily			
95	36	64	−6.84
96	40	60	−7.66
97	33	67	−7.90
98	45	55	−10.22
99	34	66	−11.40

[a] Data on the S&P 500 composite returns index and the 10-year datastream US benchmark government bond index over the period January 1990–December 1998 are used to find the optimal portfolios. Optimal portfolios consisting of US stocks and bonds are found at the point at which the risk–return trade-off equation (7) is maximised. The risk-free return is the rate on the last period's one month Treasury bill (4.47%). The VaRs for $1000 held in the portfolios are given for a daily time horizon, where the historical distribution is used to estimate the VaR.

Table 3
Optimal portfolios to meet VaR constraint under empirical distribution[a]

Confidence level (%)	Stocks (%)	Bonds (%)	Cash (%)
Daily			
95	36.00	64.00	0.00
96	35.78	53.68	10.54
97	28.63	58.12	13.25
98	30.30	37.04	32.66
99	20.55	39.89	39.56
Bi-weekly			
95	39.00	61.00	0.00
96	23.38	57.24	19.38
97	31.66	28.08	40.26
98	30.33	21.96	47.70
99	23.20	25.14	51.66
Monthly			
95	90.00	10.00	0.00
96	80.04	6.02	13.94
97	62.77	11.08	26.15
98	48.62	0.49	50.89
99	40.83	2.15	57.02

[a] Under the assumption that expected returns are distributed as in the past the optimal portfolio allocation is found such that the decision to borrow or lend is incorporated. The final optimal portfolios are found where various VaR constraints are met. These have been arbitrarily chosen to exemplify variations in individual risk–return profiles. The data used are as described in Table 2, for $1000 held in the portfolio, whereby the historical distribution at the 95% empirical level is used to estimate the VaR.

Allocating 36% in Stocks and 64% in Bonds generates a 95% VaR on the portfolio of $6.86 and of course since this is the desired VaR no borrowing or lending is required to meet the VaR constraint. If however the risk manager desires greater confidence in the probability that the initial wealth will not drop by more than the VaR level, then the VaR associated with the portfolio allocation will be greater than the VaR limit and hence results in too much risk being taken.[8] In order to meet the benchmark VaR less risk will have to be taken and hence a proportion of the initial wealth is lent at the risk free rate. This is provided in the final column where we see that the greater the confidence level, hence the lower the risk tolerance of the investor, the greater the proportion of wealth that needs to be lent at the risk free rate.[9]

The use of the empirical distribution results in the stock proportions not being a monotonic function of the confidence level. If however we assume that the future distribution of returns can be accurately proxied by the normal distribution, the only risk factor in our downside risk measure is the standard deviation of the distribution. This means that the quantile estimate is merely a multiple of standard deviations, and for short time horizons our risk measure φ in Eq. (8) depends almost entirely on the multiple of the standard deviation. This results in the risk–return trade-off being almost identical to that derived under the mean–variance framework where the Sharpe ratio is maximised. Of course since we also have the possibility of assuming different distributional assumptions, we need not constrain ourselves to optimising our portfolio according to the first two moments of the distribution only and hence are able to include the possibilities of non-normalities into asset allocation. We therefore compare the optimal allocation of assets derived using both the normal distribution and a fatter tailed distribution, the student-t, whereby we use the same sample period of data as before.

4.2. Optimal portfolio selection under alternative parametric distributions

From Fig. 1 we saw that at the 95% VaR level the assumption of normality reflects the actual risk–return trade-off fairly well. On average the assumption of normality for the future distribution of returns at the 95% level means that the risk is only slightly overestimated for a given level of return. The risk is minimised at the optimal allocation of 40% stocks and 60% bonds for daily VaR. Regardless of the confidence level, chosen for the VaR, we see that the optimal combination of risky assets is the same. Since the VaR is a multiple of the portfolio standard deviation, the assumption of normality renders the investor's attitude to risk unimportant in the optimisation process. The use of

[8] The choice of a higher confidence level will by definition result in a higher VaR.

[9] In a similar manner specifying a confidence level below that used for the optimisation the risk manager would want to take on additional risk by borrowing additional funds at the risk-free rate, and going short in the 3-month Treasury Bill.

Table 4
Optimal Portfolios to meet VaR constraint under normality and student-*t*[a]

Confidence level (%)	Normality			Student-*t*		
	Stocks (%)	Bonds (%)	Cash (%)	Stocks (%)	Bonds (%)	Cash (%)
Daily						
95	38.22	57.33	4.44	40.38	60.57	−0.95
96	35.81	53.72	10.46	36.99	55.49	7.51
97	33.24	49.86	16.90	33.33	50.00	16.67
98	30.34	45.51	24.15	29.14	43.72	27.14
99	26.67	40.01	33.32	23.73	35.59	40.68
Bi-weekly						
95	48.47	59.24	−7.71	55.59	67.94	−23.54
96	41.73	51.01	7.26	44.89	54.87	0.24
97	35.64	43.56	20.80	35.84	43.81	20.35
98	29.85	36.48	33.66	27.73	33.89	38.38
99	23.76	29.05	47.19	19.65	24.02	56.34
Monthly						
95	84.33	53.91	−38.24	190.43	121.75	−212.17
96	49.54	31.68	18.78	62.62	40.04	−2.66
97	32.88	21.02	46.10	33.31	21.30	45.39
98	22.72	14.52	62.76	19.83	12.68	67.49
99	15.28	9.77	74.96	11.43	7.31	81.26

[a] Under the assumption that expected returns are both normally and student-*t* distributed with 5 degrees of freedom, the optimal portfolio allocation is found such that the decision to borrow or lend is incorporated. The final optimal portfolios are found where various VaR constraints are met. These have been arbitrarily chosen to exemplify variations in individual's risk–return profiles. The data used are as described in Table 2, for $1000 held in the portfolio, whereby the historical distribution at the 95% empirical level is used to estimate the VaR.

longer frequency data yields an optimum of 45% stocks and 55% bonds for bi-weekly data, and 61% stocks and 39% bonds for monthly data. The maximisation also occurs at the same point as when maximising the Sharpe ratio for all types of frequencies used.[10] However through the use of VaR we are able to provide greater insight into the actual risk–return trade-off facing the investor, without having to resort to the use of specifying an individual's utility function for consumption. The exact portfolio proportions in stocks, bonds and cash to meet the 95% empirical VaR are given in Table 4 for various confidence levels and again for the various time horizons for both the normal distribution and the student-*t* with 5 degrees of freedom.

[10] Indeed for daily, bi-weekly and monthly time horizons the difference is negligible, resulting in the same optimum being found. The greater the investment time horizon however, the greater the risk-free rate of return, and hence the two risk measures will provide a different optimal points.

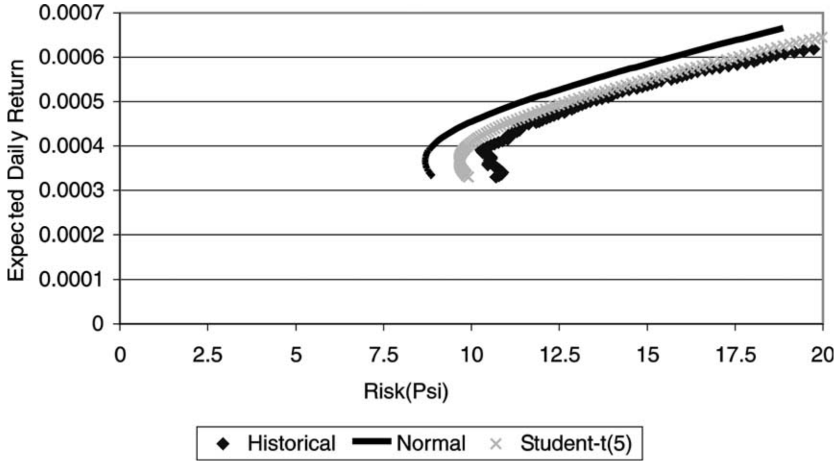

Fig. 2. Efficient VaR frontier – daily data and daily VaR at 99% confidence level.
Note: The figure presents the risk–return trade-off for portfolios of stocks and bonds whereby risk is measured by the downside risk measure φ of the portfolio at the 99% confidence level. The returns and VaR estimates are obtained using daily data on the S&P 500 composite returns index and the 10-year datastream US benchmark government bond index for the period January 1990–December 1998. We present the efficient frontier for the empirical distribution, the parametric normal approach and under the assumption of a student-*t* distribution with 5 degrees of freedom.

The non-parametric nature of the empirical distribution however, led to the changing optimum allocation of assets for various confidence levels, whereby the optimal portfolio selection resulted in a proportionally greater increase in lending to meet the desired VaR level for higher confidence levels. Under the assumption of normality with standard deviation crucial in the measure for risk this effect is not captured. Unfortunately the assumption of normality underestimates the risk–return trade-off as presented in the efficient VaR frontiers, Fig. 2. The efficient VaR frontiers at the 99% confidence level can be compared for a daily time horizon using both the normal and the empirical distributions. It appears that for a desired confidence level of 99%, for all time horizons of VaR chosen, results in too aggressive an investment strategy.

The level of risk, as measured by the empirical VaR for the portfolio, is higher for all combinations of stocks and bonds than captured by the use of standard deviation alone. The greater the deviation from normality[11] the greater the underestimation of risk as we move to higher confidence levels for the VaR. The greater probability of extreme negative returns in the empirical distribution implies greater downside risk than is captured by the measure of

[11] See, for example, Huisman et al. (1998).

standard deviation alone. The use therefore of the normal distribution to assess the risk–return trade-off will result in an incorrect allocation of assets for investors with low risk tolerance and risk managers wishing to set 99% confidence levels. The nature of the student-*t* distribution with its thin waist and fat tails gives rise to a smaller estimation of the portfolio VaR for lower confidence levels, and to a greater estimation for higher confidence levels. From Fig. 2 we see that for daily VaR it is indeed the case that at the 99% confidence level the use of normality to estimate VaR results in too high an allocation into stocks. It would therefore appear to be more appropriate to use the student-*t* distribution with 5 degrees of freedom.[12] The affect however is not so severe when a bi-weekly or a monthly time horizon is used.

As we move to higher confidence levels, for a shorter time horizon for the VaR estimation we find that it becomes more important to incorporate the additional downside risk from fat tails into the risk–return trade-off. The proportions held in the risky assets are the same as under the assumption of normality, however the portfolio risk is greater. To ensure that the final portfolio selection meets the same desired VaR level a greater proportion of the portfolio needs to be held at the risk free rate.

5. Concluding remarks

Focussing on downside risk as an alternative measure for risk in financial markets has enabled us to develop a framework for portfolio selection that moves away from the standard mean–variance approach. The measure for risk depends on a portfolio's potential loss function, itself a function of portfolio VaR. Introducing VaR into the measure for risk has the benefit of allowing the risk–return trade-off to be analysed for various associated confidence levels. Since the riskiness of an asset increases with the choice of the confidence level associated with the downside risk measure, risk becomes a function of the individual's risk aversion level. The portfolio selection problem is still to maximise expected return, however whilst minimising the downside risk as captured by VaR. This allows us to develop a very generalised framework for portfolio selection. Indeed the use of certain parametric distributions such as the normal or the student-*t* allows for a market equilibrium model to be derived, with the assumption of normality enabling the model to collapse to the

[12] The smaller the number of degrees of freedom used to parameterise the student-*t* distribution the fatter the tails of the distribution and the greater the severity of the difference between the normal distribution. Tail index estimation techniques may be adopted for the correct estimation of the degrees of freedom for the student-*t* distribution, see Huisman et al. (2001) for a robust estimator in small samples. Adopting this approach we find the use of 5 degrees of freedom throughout the empirical analysis provides consistent results.

CAPM. We illustrate just how great the impact is on the portfolio selection decision from non-normalities, alternative time horizons, and alternative risk specifications.

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