First Steps in Synthetic Computability Theory

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Abstract

Computability theory, which investigates computable functions and computable sets, lies at the foundation of computer science. Its classical presentations usually involve a fair amount of Gödel encodings which sometime obscure ingenious arguments. Consequently, there have been a number of presentations of computability theory that aimed to present the subject in an abstract and conceptually pleasing way. We build on two such approaches, Hyland's effective topos and Richman's formulation in Bishop-style constructive mathematics, and develop basic computability theory, starting from a few simple axioms. Because we want a theory that resembles ordinary mathematics as much as possible, we never speak of Turing machines and Gödel encodings, but rather use familiar concepts from set theory and topology.

Key words: synthetic computability theory, constructive mathematics.

1 Introduction

Classical presentations of the theory of computable functions and computably enumerable sets [2,7,13,16] start by describing one or more equivalent models of computation, such as Turing machines or schemata for defining partial recursive functions. Descriptions of Turing machines and numerous uses of Gödel encodings give classical computability theory its distinguishing flavor, which many "main-stream" mathematicians find unpalatable. Because of this, ingenious constructions that are unique to computability theory remain unkown to a wide audience.

One way of rectifying this situation is to present computability theory in the style of main-stream mathematics, as an abstract theory that proceeds from basic axioms, with as few explicit references to Turing machines and Gödel encodings as possible. This of course has been done before [11,6,5],

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where in most cases the axiomatizations take as the primitive notion the computation of a program on a given input, or a similar concept.

A different approach to computability theory is to work in a mathematical universe with computability built in, such as M. Hyland's *effective topos* [8] or P. Mulry's [12] *recursive topos*. In these settings, details about computations are hidden by a level of abstraction, so instead of fiddling with Turing machines and Gödel codes, one uses abstract tools, namely category theory, to achieve the desired results. How this can be done was shown by D.S. Scott, P. Mulry [12], M. Hyland [8], D. McCarty [10], G. Rosolini [15], and others.

We are going to draw on experience from the effective topos and synthetic domain theory, in particular on G. Rosolini's work [15]. However, instead of working explicitly with the topos, which requires a certain amount of knowledge of category theory *in addition* to familiarity with computability theory, we shall follow F. Richman [14] and work within ordinary (constructive) set theory enriched with few simple axioms about sets and sequences of natural numbers. Because one of our axioms contradicts Aristotle's Law of Excluded Middle, the underlying logic and set theory must be *intuitionistic*. Arguably, main-stream mathematicians consider intuitionistic mathematics to be far more exotic than computability theory. This is a historical mishap that will be amended as it gradually becomes clear that the alternative is the better choice, especially in computer science. The present paper will hopefully help spread this view.

Our goal is to develop a theory of computability *synthetically:* we work in a mathematical universe in which *all* sets and functions come equipped with intrinsic computability structure. Precisely because computability is omnipresent, we never have to speak about it—there will be no mention of Turing machines, or any other notion of computation. In the synhtetic universe, the computable functions are simply all the functions, the computably enumerable sets are all the enumerable sets, etc. So we may just speak about ordinary sets and functions and never worry about which ones are computable. For example, there is no question about what the computable real numbers should be, or how to define computability on a complicated mathematical structure. We just do "ordinary" math—in an extraordinary universe.

You may wonder how exactly such a universe is manufactured. The prime model of our theory is the effective topos. Its existence guarantees that the theory is as consistent as the rest of mathematics. In fact, the specialists will recognize it all as just clever use of the *internal language* of the effective topos. Knowledge of topos theory or the effective topos is not needed to understand synthetic computability, although familiarity with it will certainly help explain some of the axioms and constructions.

The intended audience for synthetic compuability is manifold, as there are several communities interested in computability. The constructive mathematicians should have no trouble understanding the matter, since it is written in their langauge, although the ascetic ones may find the extra axioms unacceptable. The classical computability theorists are a target audience whose approval will be the difficult to win. This is so because considerable effort is required when one first switches from classical to constructive logic, so the payoff needs to be noteworthy. A second reason is that synthetic computability has not been developed far enough to approach current research topics in computability theory. Hopefully, some experts in computability theory will be convinced that synthetic computability is a useful supplemental tool. Computer scientists tend to be more open-minded than mathematicians, so they need not worry us too much.

When an old subject is reformulated in a new way, as is the case here, success may be claimed to a lesser degree if the new formulation leads to a more elegant account, and to a larger degree when it leads to new results. The readers will be the judges of the first criterion, while it is too early to say much about the second one.

The rest of the paper is organized as follows. Section 2 contains a short introduction to constructive mathematics. Section 3 sets up the basic theory of enumerable sets and semidecidable truth values. Section 4 develops the beginnings of synthetic computability theory. A selection of standard theorems in computability is proved, among others: Single Value Theorem, Enumerability Theorem for partial recursive functions, Non-existence of a computable enumeration of total recursive functions, Projection Theorem, Post's Theorem, Existence of computably enumerable non-recursive set, Existence of inseparable sets, Existence of Kleene trees, Isomorphism of recursive versions of Cantor and Baire space, Rice's Theorem, Recursion Theorem, Berger's Branching Lemma, Rice-Shapiro, and Myhill-Shepherdson Theorems. Many of these are actually generalized, and the formulation of Recursion Theorem is new.

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2 Constructive Mathematics

The foundation of synthetic computability is constructive mathematics. This is not a philosophical decision or a matter of opinion. Aristotle's Law of Excluded Middle contradicts one of the axioms of synthetic computability, which gives us no choice but to abandon classical logic.

For the benefit of those who are not familiar with constructive mathematics we devote this section to the basic constructive setup. You may skip or just skim over it if you are familiar with the subject.

Contrary to popular belief, constructive and classical mathematics are not all that different, so it is best to just go ahead and read the text as if nothing strange happened with logic. We shall signalize and comment on the points of difference between classical and constructive mathematics. We also recommend that you take statements that blatantly oppose your mathematical intuition with excitement rather than bewilderment (remember the times when you first heard of a nowhere differentiable continuous function, a space-filling curve, infinite sets of different sizes, and strange consequences of the axiom of choice). If you are familiar with arguments in computable analysis, it may help to read each statement as if it started with the adverb "effectively", and think of every function and element as being computable. For example, the statement "for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $x < n$", should be read as "for every computable real number x we can effectively find a (computable) natural number n such that $x < n$."

Let us say a few words about the formal system that is supposed to underly our informal presentation. The text is in principle formalizable in intuitionistic higher-order logic, for example the *Zermelo set theory*, as presented by P. Taylor [18, Sect. 2.2]. We do not employ impredicativity in an essential way, and most results could be done in a weaker system based on first-order logic. Admittedly, certain details concerning the subset relation may be difficult to sort out if one wanted to figure out explicit subset inclusions. We adopt extensional equality of functions and always point out use of extra-logical axioms.

The rest of this section contains a quick introduction to constructive logic and set theory. Occasionally, we shall speak of the "computational interpretation" of this or that concept, and refer to computations and Gödel codes. Such excursions into classical computability theory are only meant to motivate and clarify what we are doing. The theory proper really does speak *only* about ordinary sets and functions, and does not rely in any way on any notion from computability theory.

2.1 The Basic Constructive Setup

We express mathematical statements with usual logical connectives $\land$, $\lor$, $\Longrightarrow$, $\Longleftrightarrow$, and quantifiers $\forall$, $\exists$. Truth and falsehood are denoted by $\top$ and $\bot$, respectively. Negation $\neg\phi$ is defined as $\phi \implies \bot$. The *unique existence* quantifier $\exists!\, x \in A \,.\, \phi(x)$ is an abbreviation for

$$\exists\, x \in A \,.\, (\phi(x) \land \forall\, y \in A \,.\, (\phi(y) \implies x = y)) \ .$$

We postpone the discussion of laws of logic until 2.4.

Our building blocks are *sets* A, B, C, ..., also called *spaces*, and *functions* f, g, h, ..., also called *maps*. A set consists of *elements* x, y, z, ..., also called *points*, which may be sets or other primitive objects. We write $x \in A$ if x is an element of A. Each set is equipped with an equality relation $=$. A word of caution to constructive mathematicians: unlike Bishop, we use $x \neq y$ as an abbreviation for $\neg(x = y)$, whereas "x is apart from y" is denoted by $x \# y$.

Every variable has a uniquely determined *domain of variation* which is a set. A variable takes on values only from its domain of variation. Therefore, when a new variable x is introduced its domain of variation A must be given, which we usually do by writing $x \in A$. Sometimes we rely on reader's ability to guess the correct domain of variation. Likewise, the universal and existential quantifiers are bounded in variation by a set, so that they are always of the form $\forall\, x \in A \,.\, \phi$ and $\exists\, x \in A \,.\, \phi$ (and never $\forall\, x \,.\, \phi$ and $\exists\, x \,.\, \phi$). In contrast, in classical set theory quantifiers and variables range over all sets, which is important for the purposes of set theorists. Our approach is closer to mainstream mathematical practice, where practically every variable is either explicitly or implicitly equipped with information about its domain of variation.

A function f has a uniquely determined *domain* and *codomain*, which are sets. We write $f : A \to B$ to indicate that A is the domain and B is the codomain of f. If $x \in A$ and $f : A \to B$ then we may apply f to x to obtain an element $f(x) \in B$. Functions $f, g : A \to B$ are equal when $f(x) = g(x)$ for all $x \in A$. Every set A has an *identity* function $1_A : A \to A$ which is characterized by $1_A(x) = x$, for all $x \in A$. The *composition* of functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$ satisfying $(g \circ f)(x) = g(f(x))$, for all $x \in A$.

An *isomorphism* is a function $f : A \to B$ which has an *inverse* $f^{-1} : B \to A$ so that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. Sets A and B are *isomorphic*, written $A \cong B$, if there is an isomorphism between them. When convenient, we shall engage in the common mathematical vice of treating isomorphic sets as equal.

A function $f : A \to B$ is *surjective*, or *onto*, when for every $y \in B$ there is $x \in A$ such that $f(x) = y$. It is *injective*, or *1–1*, when $f(x) = f(y)$ implies $x = y$, for all $x, y \in A$.

A set A is a *subset* of a set B, written $A \subseteq B$, if every element of A is also an element of B. Often we take a set A to be a subset of a set B, even though strictly speaking only an injective map $i : A \to B$ is given. In such a case we might write $A \subseteq_i B$, but only if the injection i is not an evident one. This is in accord with mathematical practice. For example, we think of the natural numbers as a subset of the real numbers, even though the natural numbers are only embedded into the real numbers by a suitable map.

We shall deal with sets of sets, which we prefer to call *families* of sets, and denote them by letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... Often a family of sets $\mathcal{A}$ is given by an *indexing* $\mathcal{A} = \{A_i \mid i \in I\}$, which is a function whose domain is an *index set* I and the values are sets A_i. We require that the sets A_i are subsets of a previously constructed ambient set, so that the indexing function can have a well-defined codomain. From the formal point of view this is a very restrictive requirement, because it prevents us from constructing certain large sets, such as the union of the iterations of powerset $\mathbb{N} \cup \mathcal{P}\mathbb{N} \cup \mathcal{P}(\mathcal{P}\mathbb{N}) \cup \mathcal{P}(\mathcal{P}(\mathcal{P}\mathbb{N})) \cup \cdots$, but in practice we shall hardly notice it.

2.2 Constructions of Sets

In this section we review the basic set-forming operations. When a new set construction is introduced, we must describe not only the constructed set, but also its equality relation.

The *cartesian product* $A \times B$ of sets A and B is the set whose elements are the *ordered pairs* $\langle x, y \rangle$ with $x \in A$ and $y \in B$. If $u \in A \times B$ then $\pi_1\, u \in A$ and $\pi_2\, u \in B$ are the *first* and *second* component of the pair u, respectively. We take ordered pairs as primitives. In classical set theory a pair $\langle x, y \rangle$ is defined to be the set $\{\{x\}, \{x, y\}\}$, which may be convenient for logicians and set theorists. We instead postulate a pairing operation $\langle -, - \rangle$ together with projections π_1 and π_2, which satisfy the axioms $\pi_1 \langle x, y \rangle = x$, $\pi_2 \langle x, y \rangle = y$, and $\langle \pi_1\, u, \pi_2\, u \rangle = u$. Ordered pairs u and v are equal when $\pi_1\, u = \pi_1\, v$ and $\pi_2\, u = \pi_2\, v$.

The *sum* $A + B$, also called *disjoint union*, of sets A and B contains elements of the form $\mathsf{inl}\, x$ with $x \in A$ and elements of the form $\mathsf{inr}\, y$ with $y \in B$. Thus, every $z \in A + B$ is either equal to $\mathsf{inl}\, x$ for a unique $x \in A$, or to $\mathsf{inr}\, y$ for a unique $y \in B$. The purpose of the tags inl and inr is to denote whether an element of $A + B$ belongs to the first or the second component. That this is necessary is clear when we consider the sum $A+A$. In classical set theory union is taken as primitive and then $A+B$ is defined as $A \times \{0\} \cup B \times \{1\}$. In our setting, union of sets only makes sense when the sets involved are all subsets of a common superset, as discussed below, which is why we prefer to take sum as the primitive operation. Elements $z, w \in A + B$ are equal when either $z = \mathsf{inl}\, x$, $w = \mathsf{inl}\, y$ and $x = y$, or $z = \mathsf{inr}\, x$, $w = \mathsf{inr}\, y$ and $x = y$. When no confusion can arise, we omit inl and inr.

The *exponential* is the set of all functions from a set A to a set B. We interchangeably use notations B^A and $A \to B$ for the exponential. Two functions $f, g : A \to B$ are considered equal when $f(x) = g(x)$ for all $x \in A$. The exponential and constructions of functions are discussed further in 2.3.

If $\phi(x)$ is a predicate on a set A, we can form its *extension* $\{x \in A \mid \phi(x)\}$, which is a subset of A containing precisely those $x \in A$ that satisfy $\phi(x)$. Every subset $S \subseteq A$ is the extension of some predicate on A, namely the predicate "x is a member of S", written simply as "$x \in S$". This establishes a bijective correspondence between predicates and subsets of A, and we shall often identify them. Elements of $\{x \in A \mid \phi(x)\}$ are equal when they are equal as elements of A. An example of a subset is the *image* of a function $f : A \to B$, which is defined as

$$\mathsf{im}(f) = \{y \in B \mid \exists\, x \in A \,.\, f(x) = y\} \;.$$

Another example is the the *complement* of $S \subseteq A$, which is the set $A \setminus S = \{x \in A \mid \neg(x \in S)\}$. Subsets $S, T \in \mathcal{P}A$ are equal when $x \in S \iff x \in T$ for all $x \in A$.

The *powerset* $\mathcal{P}A$ is the set of all subsets of A. The operations of *union* and *intersection* of a family of sets $\mathcal{F} \subseteq \mathcal{P}A$ are defined respectively by

$$\bigcup \mathcal{F} = \left\{ x \in A \;\middle|\; \exists\, S \in \mathcal{F} \,.\, x \in S \right\} \ , \quad \bigcap \mathcal{F} = \left\{ x \in A \;\middle|\; \forall\, S \in \mathcal{F} \,.\, x \in S \right\} \ .$$

As usual, a *(binary) relation* between A and B is a predicate on, or equivalently a subset of $A \times B$. A relation $R \subseteq A \times B$ is:

(i) *total* when for every $x \in A$ there exists $y \in B$ such that $R(x, y)$,

(ii) *single-valued* when for all $x \in A$ and $y, y' \in B$, if $R(x, y)$ and $R(x, y')$ then $y = y'$,

(iii) *functional* if it is total and single-valued, cf. 2.3.

There are several derived constructions of sets: quotients, products and sums. Let R be an equivalence relation on a set A. The *equivalence class* $[x]_R$ of an element $x \in A$ is the set $\left\{ y \in A \;\middle|\; R(x, y) \right\}$. The *quotient* A/R consists of all equivalence classes of R,

$$A/R = \left\{ S \in \mathcal{P}A \;\middle|\; \exists\, x \in A \,.\, S = [x]_R \right\} \ .$$

There is a canonical *quotient map* $q_R : A \to A/R$ which maps each $x \in A$ to its equivalence class $[x]_R$. The canonical quotient map is always a surjection. In fact, every surjection $f : A \to B$ is isomorphic to a quotient map, namely to $q_\sim : A \to A/{\sim}$ where $\sim$ is the equivalence relation on A defined by

$$x \sim x' \iff f(x) = f(x') \ .$$

The sets $A/{\sim}$ and B are isomorphic: an equivalence class $[x]_\sim \in A/{\sim}$ corresponds to $f(x) \in B$. Because of this we often say that B *is* a quotient of A and that f *is* a quotient map, even though it really is just isomorphic to one.

Suppose $\mathcal{A} = \left\{ A_i \;\middle|\; i \in I \right\}$ is an indexed family with $A_i \subseteq A$ for all $i \in I$. The *product* and the *sum* are the sets

$$\begin{aligned} \textstyle\prod_{i \in I} A_i &= \left\{ f \in A^I \;\middle|\; \forall\, i \in I \,.\, f(i) \in A_i \right\} \ , \\ \textstyle\sum_{i \in I} A_i &= \left\{ \langle i, x \rangle \in I \times A \;\middle|\; x \in A_i \right\} \ . \end{aligned}$$

These are generalizations of cartesian products and sums to an arbitrary family of components.

So far we have not required the existence of any sets. We now postulate the existence of *natural numbers*: there is a set $\mathbb{N}$ with an element $0 \in \mathbb{N}$, called *zero*, and a *successor* function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$, such that for every set A, $x \in A$ and $f : A \to A$ there exists a unique $h : \mathbb{N} \to A$ satisfying $h(0) = x$ and $h(\mathsf{succ}(n)) = f(h(n))$, for all $n \in \mathbb{N}$. We say that the function h is defined by *simple recursion*. From this axiom the usual arithmetical properties of natural numbers can be derived, including the induction principle.

A set A is *inhabited* if $\exists\, x \in A \,.\, x = x$ and it is *non-empty* if $\neg\forall\, x \in A \,.\, x \neq x$. Constructively, these two notions are not equivalent, for to show that a set is inhabited we must construct an element of it, whereas to demonstrate a set to be non-empty we only need to derive a contradiction from the assumption that it is empty.

A *subsingleton* is a set A with at most one element, which means that $\forall\, x, y \in A \,.\, x = y$. A *singleton* is an inhabited subsingleton. Classically, there are only two kinds of subsingletons: the singletons and the empty set. In constructive mathematics subsingletons are much more interesting, and in fact certain subsingletons play an important role in synthetic computability.

The *unit set* $\mathsf{1}$ is defined as the singleton $\mathsf{1} = \{n \in \mathbb{N} \mid n = 0\}$. We denote its only element by $\star$ rather than 0. We shall frequently work with sums of the form $\mathsf{1} + A$. In such cases, we simply write $\star$ and x, $x \in A$, instead of $\mathsf{inl}\,\star$ and $\mathsf{inr}\,x$. This amounts to assuming that $\star$ is a constant that is distinct from all elements of A.

The *empty set* $\emptyset$ is defined as the set $\emptyset = \{x \in \mathsf{1} \mid \bot\}$.

2.3 Constructions of Functions

We devote this section to ways of constructing functions. The *graph* $\Gamma(f)$ of a function $f : A \to B$ is the relation $\Gamma(f) \subseteq A \times B$ defined for $x \in A$, $y \in B$ by

$$(x, y) \in \Gamma(f) \iff f(x) = y \,.$$

It is easy to check that $\Gamma(f)$ is a functional relation. Conversely, every functional relation determines a function. This is known as the *axiom of unique choice*:

$$\forall\, R \subseteq A \times B \,.\, \big((\forall\, x \in A \,.\, \exists!\, y \in B \,.\, R(x, y)) \implies \exists\, f \in B^A \,.\, \forall\, x \in A \,.\, R(x, f(x))\big) \,.$$

It follows that there is a *unique* function whose graph is a given functional relation so that functions $A \to B$ and functional relations between A and B are in bijective correspondence.

The axiom of unique choice is useful for defining various functions. For example, the identity function 1_A could be defined as the unique function whose graph is the equality relation on A. Similarly, the composition of $f : A \to B$ and $g : B \to C$ is the unique function whose graph is $g(f(x)) = y$ with $x \in A$, $y \in C$.

The most common way of defining a function $A \to B$ is by specifying its graph in the form of an equation $y = t(x)$ with $x \in A$, $y \in B$. Here $t(x)$ is an expression in which x may appear but y may not. The unique function with this graph is denoted by $\lambda x{:}A \,.\, t(x)$. This construction of maps is called λ-*abstraction*. The β*-rule* tells us how to apply a λ-abstraction to an argument a: replace every occurrence of x in $t(x)$ with a, that is $(\lambda x{:}A \,.\, t(x))(a) = t(a)$. It is understood that the variable x is *bound* in $\lambda x{:}A \,.\, t(x)$, and that the substitution is done in a capture-avoiding way.

If $r : A \to B$ and $s : B \to A$ are maps such that $r \circ s = 1_B$ then we say that r is a *retraction* and s is a *section*. A retraction is always surjective, while a section is always injective.

Suppose we prove that for every $x \in A$ *exactly one* of conditions $\phi_1(x)$, $\dots$, $\phi_n(x)$ holds. Then we may define a function $f : A \to B$ *by cases*,

$$f(x) = \begin{cases} f_1(x) & \text{if } \phi_1(x), \\ & \vdots \\ f_n(x) & \text{if } \phi_n(x), \end{cases} \tag{1}$$

where $f_i : \{x \in A \mid \phi_i(x)\} \to B$. In particular, if $\phi(x) \lor \neg\phi(x)$ holds we may define a function

$$f(x) = \begin{cases} f_1(x) & \text{if } \phi(x), \\ f_2(x) & \text{otherwise,} \end{cases}$$

which we also write shortly as

$$f(x) = \texttt{if } \phi(x) \texttt{ then } f_1(x) \texttt{ else } f_2(x)\ .$$

Note that in constructive mathematics $\phi(x) \lor \neg\phi(x)$ does not holds generally and so it needs to be proved in each particular case.

We may also define a function by cases as in (1) when *at least one* condition $\phi_1(x)$, $\dots$, $\phi_n(x)$ holds, but then we also need to verify that $f_i(x) = f_j(x)$ whenever both $\phi_i(x)$ and $\phi_j(x)$ hold.

Lastly, if R is an equivalence relation on A and $f : A \to B$ respects R, meaning that $R(x, y)$ implies $f(x) = f(y)$, then there is a unique induced map $g : A/R \to B$ such that $g[x]_R = f(x)$ for all $x \in A$.

2.4 Laws of Logic

Our logic is *intuitionistic*. If you are familiar with it you do not need further explanation, and if you are not, an exact definition of intuitionistic rules of inference is not going to help you. Instead, we point out the basic differences between classical and intuitionistic logic, and recommend [20] for a thorough presentation of intuitionistic logic.

Intuitionistic logic is *more general* than classical logic, as it does not refute anything that classical logic validates. The correct picture to have in mind is that intuitionistic and classical logic are related to each other in the same way as non-commutative and commutative algebra—the former encompasses the later, while it also allows for new possibilities that the later does not. A mistaken belief is that intuitionistic and classical logic are like two kinds of irreconcilable theories, say spherical and Euclidean geometry.

In the setting of intuitionistic logic it is common to consider additional axioms which are incompatible with classical logic, which is just as natural as considering non-commutative operations in non-commutative algebra. One such famous example is the Fan Theorem which makes Brouwer's intuitionism anti-classical. In synthetic computability the Axiom of Enumerability is at odds with classical logic.

Most arguments in mathematical texts are intuitionistically valid. However, there are several common proof methods that are classical by nature, which we mention here so that you can avoid them in the future. Besides the Law of Excluded Middle, which states that $\phi \lor \neg\phi$ holds for any proposition ϕ, the following common logical laws are *not* generally valid in intuitionistic logic [20, 1.3]:

$$
\begin{aligned}
&\neg\neg\phi \implies \phi\,,\\
&\neg\phi \lor \neg\neg\phi\,,\\
&(\phi \implies \psi) \lor (\psi \implies \phi)\,,\\
&(\neg\psi \implies \neg\phi) \implies (\phi \implies \psi)\,,\\
&\neg(\neg\phi \land \neg\psi) \implies \phi \lor \psi\,,\\
&\neg(\neg\phi \lor \neg\psi) \implies \phi \land \psi\,,\\
&(\forall\, x \in A\,.\,\neg\neg\phi(x)) \implies \neg\neg\forall\, x \in A\,.\,\phi(x)\,,\\
&\neg\neg\exists\, x \in A\,.\,\phi(x) \implies \exists\, x \in A\,.\,\neg\neg\phi(x)\,,\\
&(\phi \implies \exists\, x \in A\,.\,\psi(x)) \implies \exists\, x \in A\,.\,(\phi \implies \psi(x))\;,
\end{aligned}
$$

where in the last formula x does not occur freely in ϕ. In practice, arguments which do not rely on any of the above rules are intuitionistically valid. The axiom of choice is discussed separately in 2.7.

2.5 The set of truth values Ω

The *truth values* are represented by sentences, i.e., propositions without free variables such as $\bot$, $\top$, and $\forall\, x \in \mathbb{R}\,.\,(x = 0 \lor x \neq 0)$. Two sentences p and q represent the same truth value when $p \iff q$. Just as a predicate $\phi(x)$ on a set A corresponds to its extension $\{x \in A \mid \phi(x)\}$, so a truth value p correspond to its extension $\{u \in \mathsf{1} \mid p\}$, where u is a dummy variable. (Henceforth we shall denote dummy variables with an underscore $\_$.) If we identify truth values with their extensions, we may define the set of truth values

$$\Omega = \mathcal{P}\mathsf{1}\;.$$

Falsehood $\bot$ and truth $\top$ are elements of Ω, represented by $\emptyset$ and $\mathsf{1}$, respectively. Are there any others? In classical logic it can be shown that every $p \in \Omega$ is equal to $\top$ or to $\bot$, because $p = \top$ is equivalent to p and $p = \bot$ is equivalent to $\neg p$, so this is just the Law of Excluded Middle: $\forall\, p \in \Omega\,.\,(p \lor \neg p)$. In intuitionistic logic it cannot be proved that every $p \in \Omega$ is equal to $\top$ or to $\bot$. But this does not mean that Ω consists of some number of elements larger than two! To explain this, we need to be more careful about what exactly it means for a set to have two elements.

Say that a set A has two elements in *the weak sense* if there are $x, y \in A$ such that $x \neq y$ and there is no $z \in A$ such that $z \neq x$ and $z \neq y$. On the other hand, say that A has two elements in *the strong sense* if there are $x, y \in A$ such that $x \neq y$ and, for any $z \in A$, either $z = x$ or $z = y$. For a classically trained mind these two senses of having two elements seem to be the same. However, constructively there is a difference, which can be explained from the computational point of view as follows. A set A has two points in the weak sense if there are computable $x \in A$ and $y \in A$ such that $x \neq y$, and there is no $z \in A$ distinct from both x and y. In contrast, a set A has two points in the strong sense if we can compute $x \in A$ and $y \in A$ such that $x \neq y$, and moreover, there is a computable procedure which for any $z \in A$ determines whether $x = z$ or $y = z$ holds. This is certainly a stronger requirement. It can be shown that any set with two elements in the strong sense is isomorphic to $\mathsf{1} + \mathsf{1}$.

Now a classical confusion arises from the fact that Ω always has two elements in the weak sense, namely $\bot$ and $\top$, but to say that it has two elements in the strong sense is to assert classical logic. Too see that there really is no third truth value, observe that $p \neq \top \land p \neq \bot$ is equivalent to $\neg p \land \neg\neg p$, a contradiction. So Ω really does have two elements in the weak sense.

The set Ω is a *complete Heyting algebra* simply because it is a powerset. As is required for a Heyting algebra, in Ω implication is the right adjoint of conjunction because $p \land q \implies r$ if, and only if, $p \implies (q \implies r)$. The supremum of a subset $S \subseteq \Omega$ is $\exists\, p \in S \,.\, p$ and the infimum is $\forall\, p \in S \,.\, p$. Negation is a *pseudo-complement*, which is to say that $\neg p$ is the largest q such that $p \land q = \bot$. For negation to be an honest complement, the Law of Excluded Middle $p \lor \neg p = \top$ would have to hold.

There are various interesting subsets of Ω. For example, we may consider the set of *decidable* truth values, which are those that satisfy the Law of Excluded Middle,

$$\mathsf{2} = \left\{ p \in \Omega \;\middle|\; p \lor \neg p \right\} .$$

Certainly $\bot$ and $\top$ are decidable truth values. Now if $p \lor \neg p$ then $p = \top$ or $p = \bot$, so we see that $\mathsf{2}$ has two elements in the strong sense. We have given it a good name, also because $\mathsf{2} = \mathsf{1} + \mathsf{1}$. When $\bot$ and $\top$ are seen as elements of $\mathsf{2}$ we prefer to denote them by 0 and 1, respectively. The set $\mathsf{2}$ is a Boolean algebra whose operations $\land$ and $\lor$ are inherited from Ω.

Another distinguished subset of Ω, which is related to classical logic, is the set of *classical* truth values:

$$\Omega_{\neg\neg} = \left\{ p \in \Omega \;\middle|\; \neg\neg p \implies p \right\} .$$

The elements of $\Omega_{\neg\neg}$ are those truth values whose truth may be established by reductio ad absurdum: if $\neg p$ implies falsehood then p holds. Because reductio ad absurdum implies classical logic we call the elements of $\Omega_{\neg\neg}$ “classical”. This is not standard terminology, but we really do want to avoid the standard awkward phrase “$\neg\neg$-stable truth values”, which expresses the fact that $\neg\neg$ is a closure operator on Ω and that $\Omega_{\neg\neg}$ is the set of its fixed points. While Ω is only a complete Heyting algebra, $\Omega_{\neg\neg}$ is also a *complete Boolean algebra.* One has to be a bit careful because not all the operations on $\Omega_{\neg\neg}$ are inherited from Ω. Implication, conjunction and infima are like in Ω, whereas binary disjunction of $p, q \in \Omega_{\neg\neg}$ and the supremum of $S \subseteq \Omega_{\neg\neg}$ are computed in Ω respectively as $\neg\neg(p \vee q)$ and $\neg\neg\exists\, p \in S \,.\, p$. As we see, double negation puts the disjunction and the supremum back into $\Omega_{\neg\neg}$.

The decidable truth values $\bot$ and $\top$ are classical, thus $\mathsf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$. In 3.3 we shall define a third subset of Ω which plays a central role in synthetic computability.

Exercise. Show that $\mathsf{2} = \Omega$ if, and only if, $\Omega_{\neg\neg} = \Omega$. What can you deduce about Ω if you assume $\mathsf{2} = \Omega_{\neg\neg}$?

2.6 Propositional Functions

A predicate $P \subseteq A$ may be equivalently expressed by a *characteristic function* $\chi_P : A \to \Omega$, defined by

$$\chi_P(x) = \left\{ \_ \in \mathsf{1} \,\middle|\, x \in P \right\} \ .$$

Conversely, a function $\xi : A \to \Omega$ corresponds to the subset

$$\left\{ x \in A \,\middle|\, \xi(x) = \top \right\} \subseteq A \ .$$

Thus we have a bijective correspondence between predicates of A, subsets of A, and *propositional functions* $A \to \Omega$. Too see this in a different way, notice that $\mathcal{P}A \cong \mathcal{P}(A \times \mathsf{1}) \cong (\mathcal{P}\mathsf{1})^A = \Omega^A$.

Propositional functions which map into a distinguished subset of Ω, such as $\mathsf{2}$ or $\Omega_{\neg\neg}$, determine special kinds of predicates. For example, a function $p : A \to \mathsf{2}$ represents a subset $S = \left\{ x \in A \,\middle|\, p(x) = \top \right\}$ which satisfies $\forall\, x \in A \,.\, (x \in S \vee x \notin S)$. Such predicates and subsets are called *decidable.* Computationally we may view a decidable predicate as one for which there exists a computable decision procedure.

A function $p : A \to \Omega_{\neg\neg}$ represents a *classical* predicate or subset of A. A subset $S \subseteq A$ is classical if, and only if, for all $x \in A$, if $x \notin A \setminus S$ then $x \in S$. A classical predicate has no computational content. If you are used just to classical logic, think of the classical predicates as the good old ones you already know.

When equality on a set has a special property, we usually give it a name. Thus a set A is *decidable* if equality on A is a decidable predicate:

$$\forall\, x, y \in A \,.\, (x = y \vee x \neq y) \ .$$

Proposition 2.1 *The following sets are decidable: natural numbers, a subset of a decidable set, the cartesian product and the sum of decidable sets.*

Proof. We prove that $\mathbb{N}$ is decidable and leave the rest as exercise. Define by recursion $e : \mathbb{N} \times \mathbb{N} \to 2$ by

$$e(0,0) = 1 \;, \qquad e(0, n+1) = 0 \;,$$
$$e(m+1, 0) = 0 \;, \qquad e(m+1, n+1) = e(m,n) \;.$$

Then $m = n$ if, and only if, $e(m,n) = 1$. □

Similarly, a set A is *classical* if equality on A is a classical predicate:

$$\forall\, x, y \in A \,.\, (\neg(x \neq y) \implies x = y) \;.$$

Every decidable set is also classical, but the converse need not hold. A subset of a classical set is classical.

Beware of a possible terminological confusion: a "decidable subset" is a subset whose membership predicate is decidable, whereas a "decidable set" is one whose equality is decidable. Thus it is easy to muddle matters when we speak about a subset with decidable equality. The same caveat stands for "classical subsets".

2.7 Axiom of Choice

In constructive mathematics the axiom of choice is handled with some care. We say that *choice holds for sets A and B*, written $\mathrm{AC}(A,B)$, when every total relation between A and B has a choice function:

$$\forall\, R \subseteq A \times B \,.\, \big((\forall\, x \in A \,.\, \exists\, y \in B \,.\, R(x,y)) \implies \exists\, f \in B^A \,.\, \forall\, x \in A \,.\, R(x, f(x))\big) \;.$$

A set A is *projective* when $\mathrm{AC}(A,B)$ holds for every set B. In classical set theory the axiom of choice states that all sets are projective. We are going to be much more restrictive because of the following computational explanation: a set A is projective when every element of A has a canonical Gödel code. Thus we would expect $\mathbb{N}$ to be projective, since in the standard Gödel coding of $\mathbb{N}$ each number is represented canonically just by itself, but we would not expect $\mathbb{N}^{\mathbb{N}}$ to be projective, because we cannot effectively choose a canonical Gödel code for each total recursive function (do you know why?).

In Bishop's constructive mathematics, the natural numbers are indeed presumed to be projective. This is known as Number Choice.

Axiom 2.2 (Number Choice) *The set of natural numbers $\mathbb{N}$ is projective.*

We shall also need Dependent Choice, which is the following generalization of Number Choice.

Axiom 2.3 (Dependent Choice) *If R is a total relation on A and $x \in A$ then there exists $f : \mathbb{N} \to A$ such that $f(0) = x$ and $R(f(n), f(n+1))$ for all $n \in \mathbb{N}$.*

The computational justification for Dependent Choice goes like this. To say that R is effectively total means that from a Gödel code m for $y \in A$ we can compute a Gödel code n for some $z \in A$ such that $R(y,z)$. Now if k is a Gödel code for x, start by defining $f(0) = x$ and $g(0) = k$. Then if $f(i) = y$ is already defined and $g(i) = m$ is a Gödel code for y, compute a Gödel code n of some z such that $R(y,z)$, then define $f(i+1) = z$ and $g(i+1) = n$. This gives us the desired choice function f.

While Number Choice suffices for most results we wish to establish, we will need projectivity of a wider class of sets.

Axiom 2.4 (Projectivity) *A classical subset of a projective set is projective.*

The Axiom of Projectivity is not generally accepted in Bishop's constructive mathematics. It is the first axiom of synthetic computability. Its computational explanation relies on the computational understanding of classical subsets: membership in a classical subset $S \subseteq A$ of a projective set A does not convey any computational content, therefore the Gödel codes of elements of S encode precisely the same information as the Gödel codes of elements of A. In fact, we may use the same Gödel codes in both cases, and since elements of A have canonical Gödel codes, so do elements of S.

All three axioms in this section are compatible with classical mathematics, since they are either special cases or simple consequences of the classical axiom of choice.

2.8 Minimization and $\mathbb{N}^+$

If a map $f : \mathbb{N} \to 2$ attains value 1 then it does so at a least argument. We would like to define a *minimization operator* $\mathsf{min} : 2^{\mathbb{N}} \to \mathbb{N}$ which assigns to each $f : \mathbb{N} \to 2$ the smallest k at which $f(k) = 1$. Of course $\mathsf{min}\, f$ is not well defined if f never attains 1. We fix this by adjoining to $\mathbb{N}$ a special *point at infinity* ∞ and let $\mathsf{min}\, f = \infty$ when $f(k) = 0$ for all $k \in \mathbb{N}$.

Let $\mathbb{N}^+$ be the set of monotone binary sequences,

$$\mathbb{N}^+ = \left\{ f \in 2^{\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N} \,.\, (f(n) = 1 \implies f(n+1) = 1) \right\} \;.$$

There is an injection $i : \mathbb{N} \to \mathbb{N}^+$ which maps n to $i(n) = \lambda k{:}\mathbb{N}\,.\,(n \le k)$. We identify $i(n)$ with n and view $\mathbb{N}$ as a subset of $\mathbb{N}^+$. The elements $i(n)$ are the *finite* points of $\mathbb{N}^+$. The constantly zero sequence $\lambda k{:}\mathbb{N}\,.\,0$ is an element of $\mathbb{N}^+$ but not of $\mathbb{N}$. We call it *infinity* and denote it by ∞. A picture of $\mathbb{N}^+$ is that of a one-point compactification of $\mathbb{N}$:

The set $\mathbb{N}^+$ is ordered by the "less than" relation $<$, defined as

$$f < g \iff \exists k \in \mathbb{N} \,.\, (f(k) = 1 \land g(k) = 0) \;.$$

It is irreflexive, asymmetric, transitive and linear, in the sense that $f < g$ implies $h < f$ or $g < h$, for all $f, g, h \in \mathbb{N}^+$. When restricted to $\mathbb{N}$, $<$ is the usual "less than" relation. For every $n \in \mathbb{N}$ we have $n < \infty$, as expected. If $n \in \mathbb{N}$ and $g \in \mathbb{N}^+$, the proposition $i(n) < g$ is decidable because it is equivalent to $g(n) = 0$.

Similarly, $\leq$ on $\mathbb{N}^+$ is defined by

$$f \leq g \iff \neg(g < f) \iff \forall\, k \in \mathbb{N}\,.\,(f(k) = 0 \implies g(k) = 0)\ .$$

The *minimization operator* $\mathsf{min} : \mathsf{2}^{\mathbb{N}} \to \mathbb{N}^+$ may now be defined as follows:

$$\mathsf{min}\, f = \lambda k\!:\!\mathbb{N}\,.\ (\exists\, j \leq k\,.\, f(j) = 1)\ .$$

We write $\mathsf{min}_k\, t(k)$ for $\mathsf{min}\,(\lambda k\!:\!\mathbb{N}\,.\,t(k))$. It is always the case that $f(k) = 0$ for all $k < \mathsf{min}\, f$, and if $\mathsf{min}\, f \in \mathbb{N}$ then $f(\mathsf{min}\, f) = 1$. It also holds that if $f(k) = 1$ for some $k \in \mathbb{N}$ then $\mathsf{min}\, f \leq k$. Finally, observe that min is a retraction whose section is the inclusion $i : \mathbb{N}^+ \to \mathsf{2}^{\mathbb{N}}$.

3 Enumerable and Semidecidable Sets

We now finally embark properly on the subject of computability theory. Unlike in most classical treatments of computability theory, we do not start with partial recursive functions but rather with computably enumerable sets. This is so because in our settings the computably enumerable sets are simply the enumerable sets, while some effort is needed to introduce the partial recursive functions.

3.1 Finite lists and finite sets

As a warm up, we first review the basic constructive theory of finite sequences and finite sets. Computationally speaking, a set is finite if we can compute a finite sequence of Gödel codes of its elements, where some elements may be represented more than once.

For a natural number $n \in \mathbb{N}$, define the set $\{1, \ldots, n\} = \{k \in \mathbb{N} \mid k \leq n\}$. A *(finite) list*, or a *finite sequence* of elements in A is a map $\ell : \{1, \ldots, n\} \to A$ for some $n \in \mathbb{N}$. The number n is called the *length* of ℓ and is denoted by $|\ell|$. We write a list ℓ by $[\ell(1), \ldots, \ell(n)]$. Given $x \in A$ we may form a new list $x{::}\ell$ whose first element is x followed by elements of ℓ. Every list is either empty or of the form $x{::}\ell$ for unique x and ℓ. The set of all finite sequences of elements of A is

$$\mathsf{Seq}\, A = \sum_{n \in \mathbb{N}} A^{\{1,\ldots,n\}}\ .$$

A set A is *finite* if there exists $n \in \mathbb{N}$ and a surjection $e : \{1, \ldots, n\} \to A$, called a *listing* of A. The collection of all finite subsets of A is denoted by $\mathcal{P}_{\text{fin}} A$. There is a quotient map $\mathsf{Seq}\, A \to \mathcal{P}_{\text{fin}} A$ which assigns to a list ℓ its image $\mathsf{im}(\ell) = \{x \in A \mid \exists\, k \in \{1, \ldots, |\ell|\}\,.\, x = \ell(k)\}$. A set is *subfinite* if it is a subset of a finite set.

Our definition of finite sets is equivalent to that of *Kuratowski finite set:*

Proposition 3.1 *The set $\mathcal{P}_{\text{fin}}A$ is the join-semilattice generated by A, which means that it is the smallest family $\mathcal{F} \subseteq \mathcal{P}A$ satisfying the following conditions:*

(i) $\emptyset \in \mathcal{F}$,

(ii) $\{x\} \in \mathcal{F}$ *for all* $x \in A$, *and*

(iii) *if* $S, T \in \mathcal{F}$ *then* $S \cup T \in \mathcal{F}$.

Proof. Exercise. □

The following is a useful observation: a finite set is either empty or inhabited. For suppose $e : \{1, \dots, n\} \to A$ is a listing of A. If $n = 0$ then A is empty, and if $n \neq 0$ then A is inhabited by $e(1)$.

Proposition 3.2 *A quotient of a finite set is finite.*

Proof. If S is listed by $[x_1, \dots, x_n]$ and $q : S \to T$ is a surjection, then $[q(x_1), \dots, q(x_n)]$ is a listing of T. □

Let us dispel two common misconceptions about finite sets which sometimes carry over from classical mathematics. Firstly, it *cannot* be shown constructively that every finite set has a well-defined size which is a natural number. Suppose we had a "size" operation which assigned equal integers to isomorphic finite sets and different integers to non-isomorphic finite sets. Then for any $p \in \Omega$, we could look at the size of the set $\mathsf{2}/{\sim}$ where $\sim$ is the equivalence relation

$$x \sim y \iff ((x = y) \lor p) .$$

The size of $\mathsf{2}/{\sim}$ would equal the size of $\mathsf{1}$ if, and only if, p were true. But this would imply the Law of Excluded middle, since the sizes of $\mathsf{2}/{\sim}$ and $\mathsf{1}$ would be integers, which are either equal or not.

Secondly, in constructive mathematics a subset of a finite set need not be finite. To see this, suppose every subfinite set were finite. Then for any $p \in \Omega$ the set $\{\_ \in \mathsf{1} \mid p\}$ would be finite, therefore either empty or inhabited. Again this would imply the Law of Excluded Middle: if the set is inhabited then p holds, otherwise $\neg p$ holds.

In a listing of a finite set some elements may be repeated. Which finite sets can be listed without repetition?

Proposition 3.3 *A finite set may be listed without repetitions if, and only if, it is decidable.*

Proof. If $e : \{1, \dots, n\} \to A$ is a listing without repetitions, then it is surjective and injective, hence an isomorphism. The set A is decidable because it is isomorphic to the decidable set $\{1, \dots, n\}$. To obtain the converse, suppose A is decidable. First define a function f which removes duplicates from a list of elements of A:

$$f\,[\,] = [\,]\,,$$
$$f\,[x_0, x_1, \ldots, x_n] = \begin{cases} f[x_1, \ldots, x_n] & \text{if } x_0 = x_i \text{ for some } 1 \le i \le n, \\ x_0 {::} f[x_1, \ldots, x_n] & \text{otherwise.} \end{cases}$$

The function f is well defined because we can decide whether x_0 is equal to one of x_1, …, x_n. For any listing ℓ of A, $f(\ell)$ is a listing of A without repetitions. □

Proposition 3.4 *If A is decidable then so are $\mathsf{Seq}\,A$ and $\mathcal{P}_{\text{fin}}A$.*

Proof. Suppose A is decidable. For $p, q \in \mathsf{Seq}\,A$, $p = q$ if, and only if,

$$|p| = |q| \wedge \forall\, k \in \{1, \ldots, |p|\}\;.\, p(k) = q(k)\;,$$

which is a decidable proposition. For $x \in A$ and $p \in \mathsf{Seq}\,A$, we may decide whether $x \in \mathsf{im}(p)$ simply by checking $x = p(k)$ for $k \in \{1, \ldots, 1, |p|\}$. Thus also $\mathsf{im}(p) \subseteq \mathsf{im}(q)$ is decidable for $p, q \in \mathsf{Seq}\,A$, since it is equivalent to $\forall\, k \in \{1, \ldots, |p|\}\;.\, p(k) \in \mathsf{im}(q)$. To conclude that $\mathcal{P}_{\text{fin}}A$ is decidable, consider any $S, T \in \mathcal{P}_{\text{fin}}A$ listed by $p, q \in \mathsf{Seq}\,A$ and observe that $S = T$ if, and only if, $\mathsf{im}(p) \subseteq \mathsf{im}(q)$ and $\mathsf{im}(q) \subseteq \mathsf{im}(p)$. □

3.2 Enumerable Sets

The computably enumerable sets, formerly known as the recursively enumerable sets, play a central role in classical computability theory. They are a basic concept in the synthetic theory as well, except that they are now just “ordinary” enumerable sets.

Definition 3.5 A set A is *enumerable*, or *countable*, if there exists a surjection $e : \mathbb{N} \to \mathbf{1} + A$, called an *enumeration* of A. A set is *subenumerable*, or *subcountable*, if it is a subset of an enumerable set.

The reason for mapping $\mathbb{N}$ onto $\mathbf{1} + A$ rather than onto A is that we also want to include the empty set among the enumerable ones. The role of the special element $\star \in \mathbf{1}$ is to enumerate nothing, so that the empty set may be enumerated by $\star$, $\star$, $\star$, …

First properties of enumerable sets are collected in the following proposition.

Proposition 3.6

(i) *A set A is inhabited and enumerable if, and only if, there exists a surjection $e : \mathbb{N} \to A$.*

(ii) *A quotient of an enumerable set is enumerable.*

(iii) *A finite set is enumerable.*

Proof.

(i) If A is inhabited by $x \in A$ and enumerated by $e : \mathbb{N} \to \mathbf{1} + A$, we may enumerate it with a surjection $e' : \mathbb{N} \to A$ defined by $e'(n) = \texttt{if}\ e(n) = \star\ \texttt{then}\ x\ \texttt{else}\ e(n)$. Conversely, if there is a surjection $e : \mathbb{N} \to A$ then A is enumerated by e and inhabited by $e(0)$.

(ii) If $f : A \to B$ is onto and A is enumerated by $e : \mathbb{N} \to \mathbf{1} + A$ then B is enumerated by $e'(n) = \texttt{if}\ e(n) = \star\ \texttt{then}\ \star\ \texttt{else}\ f(e(n))$.

(iii) A finite set is a quotient of the enumerable set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$. □

We say that $e : \mathbb{N} \to \mathbf{1} + A$ enumerates A *without repetitions* when e is an enumeration and, for all $n, m \in \mathbb{N}$, $e(n) = e(m) \neq \star$ implies $n = m$.

Proposition 3.7 *A set can be enumerated without repetitions if, and only if, it is enumerable and decidable.*

Proof. Suppose $e : \mathbb{N} \to \mathbf{1} + A$ is an enumeration without repetitions, and let $x, y \in A$. There exist unique $m, n \in \mathbb{N}$ such that $e(m) = x$ and $e(n) = y$. Then we have $x = y$ if, and only if, $m = n$. So A has decidable equality because $\mathbb{N}$ does. Conversely, suppose A has decidable equality and $e : \mathbb{N} \to \mathbf{1} + A$ enumerates it. Define a new enumeration $e' : \mathbb{N} \to \mathbf{1} + A$ without repetitions by

$$e'(n) = \begin{cases} e(n) & \text{if } \forall\, k < n \,.\, e(k) \neq e(n), \\ \star & \text{otherwise.} \end{cases}$$

□

Corollary 3.8 *Every enumerable subset of* $\mathbb{N}$ *can be enumerated without repetitions.*

Proof. Every subset of $\mathbb{N}$ is decidable. □

We say that a set A *contains an infinite sequence* if there exists an injection $a : \mathbb{N} \to A$. When $A \subseteq \mathbb{N}$, such an injection may always be replaced by a strictly increasing one.

Proposition 3.9 *An inhabited enumerable subset of* $\mathbb{N}$ *may be enumerated in a strictly increasing order if, and only if, it is a decidable subset of* $\mathbb{N}$ *and it contains an infinite sequence.*

Proof. If $e : \mathbb{N} \to A$ is a strictly increasing enumeration of $A \subseteq \mathbb{N}$ then it is an infinite sequence. The set A is decidable because, for any $n \in \mathbb{N}$, $n \in A$ precisely in case that $\exists\, m \leq n \,.\, e(m) = n$. Conversely, suppose A is a decidable subset of $\mathbb{N}$ and let $a : \mathbb{N} \to A$ be a strictly increasing sequence in A. A strictly increasing enumeration of A may be defined by $e(n) = \mathsf{min}_k\,(k \in A \wedge e(n) < k \leq a(n))$. □

Proposition 3.10 *A decidable enumerable inhabited set A is a retract of* $\mathbb{N}$*. Furthermore, if A contains an infinite sequence then it is isomorphic to* $\mathbb{N}$*.*

Proof. Let $e : \mathbb{N} \to A$ be an enumeration of A. Define $f : A \to \mathbb{N}$ by $f(x) = \mathsf{min}_k\,(e(k) = x)$, which is a valid definition because A is decidable and e is surjective. Clearly we have $e(f(x)) = x$ for every $x \in A$, which proves that A is a retract of $\mathbb{N}$. Furthermore, if $A \subseteq_f \mathbb{N}$ contains an infinite sequence then by Proposition 3.9 it is enumerated in a strictly increasing order by some $e' : \mathbb{N} \to A$. Because e' is surjective and injective, it is an isomorphism. □

Corollary 3.11 *The following sets are isomorphic to* $\mathbb{N}$*:*

(i) *the set of k-tuples $\mathbb{N}^k$, with $k \geq 1$,*

(ii) *the set of finite sequences $\mathsf{Seq}\,\mathbb{N}$,*

(iii) *the family of finite subsets $\mathcal{P}_{\text{fin}}\mathbb{N}$.*

Proof. These sets are decidable and contain infinite sequences. We only need to check that they are enumerable. For $n \geq 1$ and a prime number p, let $r(n,p) = \max\left\{k \in \mathbb{N} \mid p^k \text{ divides } n\right\}$. Let p_i be the i-th prime number. We may take as the n-th k-tuple $\langle r(n,p_1), \ldots, r(n,p_2)\rangle$, as the n-th list $\langle r(n,p_2), \ldots, r(n,p_{p(n,p_1)})\rangle$, and as the n-th finite set the one enumerated by the n-th list. □

The preceding proposition tells us that we may enumerate the elements of a set with k-tuples, finite sequences or finite sets of numbers. We shall do so whenever convenient. We next investigate operations under which enumerable sets are preserved.

Proposition 3.12

(i) *The cartesian product of two enumerable sets is enumerable.*

(ii) *The dependent sum of an enumerable family of enumerable sets is enumerable.*

(iii) *The union of an enumerable family of enumerable sets is enumerable.*

(iv) *The intersection of two enumerable subsets of a decidable set is enumerable.*

(v) *A decidable subset of an enumerable subset is enumerable.*

(vi) *The finite sequences of an enumerable set form an enumerable set.*

(vii) *The finite subsets of an enumerable set form an enumerable family.*

Proof. These are all easily proved. For illustration we prove the second claim in detail. Let I be enumerable and $\left\{A_i \mid i \in I\right\}$ an indexing of enumerable sets $A_i \subseteq A$. We need to show that $S = \sum_{i \in I} A_i$ is enumerable. If $e : \mathbb{N} \to 1 + I$ is an enumeration of I then for every $n \in \mathbb{N}$, either $e(n) = \star$ or there is an enumeration of $A_{e(n)}$. By Countable Choice there is a function $f : \mathbb{N} \to 1 + (1 + A)^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, either $e(n) = \star$ and $f(n) = \star$, or $f(n)$ is an enumeration of $A_{e(n)}$. The set S is enumerated by the function

$e' : \mathbb{N} \times \mathbb{N} \to 1 + S$, defined by

$$e'\langle m, n\rangle = \begin{cases} \langle e(m), f(m)(n)\rangle & \text{if } e(m) \neq \star \text{ and } f(m) \neq \star, \\ \star & \text{otherwise.} \end{cases}$$

To see that e' is a surjection, consider any $\langle i, x\rangle \in S$. There is $m \in \mathbb{N}$ such that $e(m) = i$. Then $f(m)$ enumerates A_i, so there is $n \in \mathbb{N}$ such that $f(m)(n) = x$, hence $e'\langle m, n\rangle = \langle i, x\rangle$. □

We have exhibited some enumerable sets, but we still do not know whether there are any sets that are *not* enumerable. Let us show that *Cantor space* $2^{\mathbb{N}}$ and *Baire space* $\mathbb{N}^{\mathbb{N}}$ are not enumerable. Recall that a set A has the *fixed point property* if every map $f : A \to A$ has a fixed point, which is an element $x \in A$ such that $f(x) = x$. We use Lawvere's formulation of Cantor's argument.

Proposition 3.13 (Lawvere) *If $e : A \to B^A$ is a surjection then B has the fixed point property.*

Proof. Given $f : B \to B$, there is $x \in A$ such that $e(x) = \lambda y{:}A\,.\,f(e(y)(y))$ because e is surjective. Then $e(x)(x) = f(e(x)(x))$, hence $e(x)(x)$ is a fixed point of f. □

In passing we prove a famous theorem by Cantor.

Corollary 3.14 (Cantor's Theorem) *There is no surjection $A \twoheadrightarrow \mathcal{P}A$.*

Proof. If there were a surjection $A \twoheadrightarrow \mathcal{P}A = \Omega^A$ then Ω would have the fixed point property, but it does not because negation $\neg : \Omega \to \Omega$ does not have a fixed point. Indeed, if $p = \neg p$ then $p = p \wedge p = p \wedge \neg p = \bot$, but then $\bot = \neg\bot = \top$, contradiction. □

Corollary 3.15 Cantor space $2^{\mathbb{N}}$ *and* Baire space $\mathbb{N}^{\mathbb{N}}$ *are* not *enumerable.*

Proof. The sets 2 and $\mathbb{N}$ do not have the fixed point property. □

The preceding corollary is the analogue of recursion-theoretic facts that total recursive functions and recursively decidable sets cannot be effectively enumerated.

Corollary 3.16 *If a set has the fixed point property and contains two distinct points then it does* not *have decidable equality.*

Proof. Suppose $x, y \in A$ are distinct elements of a set A with decidable equality. Then the map $f : A \to A$ defined by $f(z) = \texttt{if } x = z \texttt{ then } y \texttt{ else } x$ does not have a fixed point. □

Of particular interest to us is the set of enumerable subsets of $\mathbb{N}$, which we denote by $\mathcal{E}$:

$$\mathcal{E} = \left\{ A \in \mathcal{P}\mathbb{N} \;\middle|\; A \text{ is enumerable} \right\} .$$

Theorem 3.17 *The family $\mathcal{E}$ is the least family $\mathcal{F} \subseteq \mathcal{P}\mathbb{N}$ such that:*

(i) $\emptyset \in \mathcal{F}$ *and* $\mathbb{N} \in \mathcal{F}$,

(ii) $\{n\} \in \mathcal{F}$ *for every* $n \in \mathbb{N}$,

(iii) *if* $A, B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$, *and*

(iv) *if* $\mathcal{A} \subseteq \mathcal{F}$ *is an enumerable family then* $\bigcup \mathcal{A} \in \mathcal{F}$.

Proof. The empty set, $\mathbb{N}$ and singletons are clearly enumerable. That $\mathcal{E}$ is closed under binary intersections and enumerable unions was proved in Proposition 3.12(iv) and 3.12(iii).

Now suppose $\mathcal{F} \subseteq \mathcal{P}\mathbb{N}$ satisfies the above four conditions, and let $e : \mathbb{N} \to 1 + A$ be an enumeration of $A \subseteq \mathbb{N}$. Then $A \in \mathcal{F}$ because A is the union of the enumerable family $\mathcal{A} = \{A_i \in i \in \mathbb{N}\}$ of enumerable sets $A_i = \{m \in \mathbb{N} \mid m = e(i)\}$. □

The family $\mathcal{E}$ is like the topology on $\mathbb{N}$ generated by singletons, i.e., the discrete topology, except that it is closed under enumerable unions rather than arbitrary ones. We shall devote much more attention to this topological view of enumerable sets in 3.5.

Recall that the *projection* of a subset $S \subseteq A \times B$ is the set

$$\{x \in A \mid \exists y \in B \,.\, \langle x, y\rangle \in S\} \ .$$

Theorem 3.18 (Projection Theorem) *A subset of* $\mathbb{N}$ *is enumerable if, and only if, it is the projection of a decidable subset of* $\mathbb{N} \times \mathbb{N}$.

Proof. Assume first that A is enumerated by $e : \mathbb{N} \to 1 + A$. For the set S we simply take the graph of e,

$$S = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid e(m) = n\} \ ,$$

which is a decidable set because $\mathbb{N}$ has decidable equality. Conversely, suppose $S \subseteq \mathbb{N} \times \mathbb{N}$ is decidable and let $A = \{m \in \mathbb{N} \mid \exists n \in \mathbb{N} \,.\, \langle m, n\rangle \in S\}$. Define a map $e : \mathbb{N} \times \mathbb{N} \to 1 + \mathbb{N}$ by

$$e\langle i, j\rangle = \begin{cases} i & \text{if } \langle i, j\rangle \in S, \\ \star & \text{otherwise.} \end{cases}$$

Then e is well defined because S is decidable, and it obviously enumerates A. □

3.3 The Semidecidable Truth Values

The computably enumerable sets are also known as the "semidecidable" sets. In this section we show that also in synthetic computability the enumerable sets are semidecidable in a precise sense: we find a set $\Sigma \subseteq \Omega$ of "semidecidable" truth values such that $\mathcal{E} = \Sigma^{\mathbb{N}}$.

Recall that by the Projection Theorem every $A \in \mathcal{E}$ is the projection of a decidable subset $D \subseteq \mathbb{N} \times \mathbb{N}$. Let $d : \mathbb{N} \times \mathbb{N} \to 2$ be the characteristic function of D. Then the characteristic function $\chi_A : \mathbb{N} \to \Omega$ of A is

$$\chi_A(m) = (\exists\, n \in \mathbb{N} \,.\, d\langle m, n\rangle) \;.$$

For a fixed $m \in \mathbb{N}$, the truth value $\chi_A(m)$ is of the form $\exists\, n \in \mathbb{N} \,.\, f(n)$ where $f : \mathbb{N} \to 2$ is $f(n) = d\langle m, n\rangle$. So we may define the set of *semidecidable* truth values

$$\Sigma = \left\{ p \in \Omega \;\middle|\; \exists\, f \in 2^{\mathbb{N}} \,.\, (p \iff (\exists\, n \in \mathbb{N} \,.\, f(n))) \right\} \;.$$

As a first observation, note that $2 \subseteq \Sigma$ because $0 \iff \exists\, n \in \mathbb{N} \,.\, (\lambda k : \mathbb{N} \,.\, 0)(n) = 1$ and $1 \iff \exists\, n \in \mathbb{N} \,.\, (\lambda k : \mathbb{N} \,.\, 1)(n) = 1$. Therefore the decidable truth values 0 and 1 are semidecidable, as expected. The set Σ has two elements in the weak sense. Later we shall prove that it has two elements in the strong sense if, and only if, the Limited Principle of Omniscience holds.

In general, we define a *semidecidable* subset $S \subseteq A$ to be one whose characteristic map $\chi_S : A \to \Omega$ maps into Σ, that is $(x \in S) \in \Sigma$ for all $x \in A$.

Proposition 3.19 *A subset of $\mathbb{N}$ is enumerable if, and only if, it is semidecidable.*

Proof. We have essentially verified this above, but let as spell out more details. For any $A \in \mathcal{E}$ there is, by Projection Theorem, a map $d : \mathbb{N} \times \mathbb{N} \to 2$ such that $m \in A \iff \exists\, n \in \mathbb{N} \,.\, d\langle m, n\rangle = 1$. Thus the characteristic map $\chi_A : \mathbb{N} \to \Omega$ is

$$\chi_A(m) = \exists\, n \in \mathbb{N} \,.\, d\langle m, n\rangle = 1 \;,$$

which is clearly a map into Σ. Conversely, suppose $A \subseteq \mathbb{N}$ and $\chi_A : \mathbb{N} \to \Sigma$. By Countable Choice there exists a sequence $f : \mathbb{N} \to 2^{\mathbb{N}}$ such that $g(m) = \exists\, n \in \mathbb{N} \,.\, f(m)(n) = 1$, for all $m \in \mathbb{N}$. Now A is enumerable because it is the projection of the decidable set $\left\{ \langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \;\middle|\; f(m)(n) = 1 \right\}$. $\square$

The definition of Σ is well known in *syntethic domain theory* and appears in [15]. It is a *dominance*, which means that it satisfies $\top \in \Sigma$ and Rosolini's dominance axiom

$$\forall\, p \in \Sigma \,.\, \forall\, q \in \Omega \,.\, ((p \implies (q \in \Sigma)) \implies (p \wedge q) \in \Sigma) \;.$$

Too see this, suppose $p \iff \exists\, n \in \mathbb{N} \,.\, f(n)$ for $f : \mathbb{N} \to 2$ and $p \implies (q \in \Sigma)$. For every $n \in \mathbb{N}$,

$$f(n) = 0 \vee (f(n) = 1 \wedge \exists\, g \in 2^{\mathbb{N}} \,.\, (q \iff \exists\, n \in \mathbb{N} \,.\, g(n))) \;.$$

By Countable Choice there exists $h : \mathbb{N} \to 1 + 2^{\mathbb{N}}$ such that, for all $m \in \mathbb{N}$, either $f(m) = 0$ and $h(m) = \star$, or $f(m) = 1$ and $q \iff \exists\, n \in \mathbb{N} \,.\, h(m)(n) = 1$. Define $k : \mathbb{N} \times \mathbb{N} \to 2$ by

$$k(m, n) = \begin{cases} 0 & \text{if } f(m) = 0, \\ h(m)(n) & \text{if } f(m) = 1. \end{cases}$$

It is easy to see that $p \wedge q \iff \exists\, \langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \,.\, k(m, n) = 1$.

We next investigate the order-theoretic structure of Σ, which inherits a partial order from the complete Heyting algebra Ω. Recall that a lattice is a poset L with least and greatest elements, and binary infima and suprema. A lattice is a *frame* if arbitrary suprema exist and finite infima distribute over arbitrary suprema.

Definition 3.20 A *σ-frame* is a non-trivial lattice L in which suprema of enumerable sets exist and binary infima distribute over enumerable suprema. A *morphism* between σ-frames is a map which preserves the lattice structure and enumerable suprema.

The definition of σ-frames was given by Rosolini [15] who called them "σ-algebras". We prefer not to call them σ-algebras in order to avoid confusion with measure-theoretic σ-algebras.

Proposition 3.21 *Σ is the initial σ-frame: for every σ-frame L there exists a unique morphism $\Sigma \to L$.*

Proof. A morphism $\phi : \Sigma \to L$, if it exists, must map $\top$ to 1_L and $\bot$ to 0_L. But then it is already determined on all of Σ because every element $p \in \Sigma$ is a countable union of $\top$'s and $\bot$'s, and ϕ preserves countable joins. Indeed, $p \iff \exists\, n \in \mathbb{N} \,.\, f(n) = 1$ for some $f : \mathbb{N} \to 2$, therefore $p = \bigvee_{n \in \mathbb{N}} f(n)$ and it must be the case that

$$\phi(\exists\, n \in \mathbb{N} \,.\, f(n)) = \bigvee_{n \in \mathbb{N}} (\mathtt{if}\ f(n)\ \mathtt{then}\ 1_L\ \mathtt{else}\ 0_L)\ .$$

So there is at most one morphism $\Sigma \to L$, and there actually is one, namely the one just described. □

Proposition 3.22 *Σ is the smallest subset of Ω which contains $\top$ and is closed under enumerable suprema.*

Proof. We already know that $\top \in \Sigma$. Suppose $e : \mathbb{N} \to \mathbf{1} + S$ is an enumeration of $S \subseteq \Sigma$. Then $\exists\, p \in S \,.\, p$ is equivalent to $\exists\, n \in \mathbb{N} \,.\, e(n) = \top$, which is semidecidable because it is equivalent to $\exists\, \langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \,.\, f(m, n) = 1$ with $f : \mathbb{N} \times \mathbb{N} \to 2$ defined as follows. By Countable Choice there exists $g : \mathbb{N} \to \mathbf{1} + 2^{\mathbb{N}}$ such that, for all $m \in \mathbb{N}$, if $e(m) = \star$ then $g(m) = \star$ and if $e(m) \in \Sigma$ then $g(m) \in 2^{\mathbb{N}}$ and $e(m) \iff \exists\, n \in \mathbb{N} \,.\, g(m)(n) = 1$. Define

$$f(m, n) = \mathtt{if}\ g(m) = \star\ \mathtt{then}\ 0\ \mathtt{else}\ g(m)(n)\ .$$

Conversely, suppose $F \subseteq \Omega$ contains $\top$ and is closed under enumerable suprema. Let $p \in \Sigma$ where $p \iff \exists\, n \in \mathbb{N} \,.\, f(n) = 1$. Then $p \iff \exists\, q \in S \,.\, q$ where $S = \{r \in \Omega \mid (r = \top) \wedge \exists\, n \in \mathbb{N} \,.\, f(n) = 1\}$. The set S is enumerated by $e(n) = \mathtt{if}\ f(n) = 1\ \mathtt{then}\ \top\ \mathtt{else}\ \star$, therefore $p \in F$, as required. □

We mention one more characterization of Σ.

Proposition 3.23 *Σ is a quotient of $\mathbb{N}^+$ via the map $q : \mathbb{N}^+ \to \Sigma$ defined by $q(x) = (x < \infty)$.*

Proof. Recall that $x \in \mathbb{N}^+$ is a binary sequence which is smaller than ∞ when it contains a 1. So $x < \infty$ is equivalent to $\exists\, n \in \mathbb{N} \,.\, x(n) = 1$, which is semidecidable. The map q is surjective because any $p \in \Sigma$ with $p \iff \exists\, n \in \mathbb{N} \,.\, f(n) = 1$ is equivalent to $x < \infty$ where $x(n) = (\exists\, k \leq n \,.\, f(k) = 1)$. □

If we insisted on a predicative definition of Σ (one that does not refer to Ω), we could use Proposition 3.21 or 3.23 as a definition.

3.4 Markov Principle

If $a_0, a_1, a_2, \ldots$ is a sequence of zeros and ones, not all of which are zeros, must there be a one in the sequence? An affirmative answer is known as *Markov Principle* and has several equivalent forms.

Proposition 3.24 *The following are equivalent:*

(i) *Markov Principle: for every $a : \mathbb{N} \to 2$,*

$$\neg(\forall\, n \in \mathbb{N} \,.\, a_n = 0) \implies \exists\, n \in \mathbb{N} \,.\, a_n = 1 \,.$$

(ii) *Semidecidable truth values are classical, $\Sigma \subseteq \Omega_{\neg\neg}$.*

(iii) *Semidecidable subsets are classical.*

(iv) *Semidecidable subsets of $\mathbb{N}$ are classical.*

Proof. Because $\neg(\forall\, n \in \mathbb{N} \,.\, a_n = 0)$ is equivalent to $\neg\neg(\exists\, n \in \mathbb{N} \,.\, a_n = 1)$, Markov Principle is equivalent to $\forall\, p \in \Sigma \,.\, (\neg\neg p \implies p)$, which is the second statement. The second statement may be rephrased as $\Sigma \subseteq \Omega_{\neg\neg}$, from which it is clear that the characteristic map $\chi_A : A \to \Sigma$ of a semidecidable subset $S \subseteq A$ also characterizes S as a classical subset, as a map $\chi_A : A \to \Sigma \hookrightarrow \Omega_{\neg\neg}$. This proves the third statement from the second one. The fourth statement is a special case of the third one. Finally, the fourth statement implies the second one: if $p \in \Sigma$ then $S = \{n \in \mathbb{N} \mid p\}$ is semidecidable, hence classical, so p is classical because it is equivalent to $0 \in S$. □

Exercise. Show that Markov Principle is equivalent to: for all $x \in \mathbb{N}^+$, if $x \neq \infty$ then $x \in \mathbb{N}$.

While it may seem intuitively clear that Markov Principle holds, it cannot be proved constructively. A number of results in computability rely on its validity. Therfore, we accept it as an axiom.

Axiom 3.25 (Markov Principle) *A binary sequence which is not constantly 0 contains a 1.*

Proposition 3.26 *Equality and the partial order on Σ are classical.*

Proof. Because $\Sigma \subseteq \Omega_{\neg\neg}$ and $\Omega_{\neg\neg}$ has classical equality and partial order, Σ does as well. $\square$

The following is a useful consequence of Markov Principle.

Lemma 3.27 *For any classical predicate $\psi : \Sigma \to \Omega_{\neg\neg}$, $(\forall\, p \in \Sigma \,.\, \psi(p)) \iff \psi(\bot) \wedge \psi(\top)$.*

Proof. One direction is obvious. For the other, suppose $\psi(\bot)$ and $\psi(\top)$. Then, for any $p \in \Sigma$, $(p = \top \vee p = \bot) \implies \psi(p)$, hence $\neg\neg(p = \top \vee p = \bot) \implies \neg\neg\psi(p)$. But $\neg\neg(p = \top \vee p = \bot) = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top$ and $\neg\neg\psi(p) = \psi(p)$, therefore $\psi(p)$ for every $p \in \Sigma$. $\square$

Every book on computability theory contains (the special case $A = \mathbb{N}$ of) the following theorem.

Theorem 3.28 (Post) *A subset of a set A is decidable if, and only if, it and its complement are semidecidable.*

Proof. The theorem may be rephrased in terms of truth values: a truth value $p \in \Omega$ is decidable if, and only if, p and $\neg p$ are semidecidable. Obviously since $2 \subseteq \Sigma$, a decidable truth value and its complement are semidecidable. Conversely, suppose $p \in \Sigma$ and $\neg p \in \Sigma$. Then by Markov Principle $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore $p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top$, hence $p \in 2$. $\square$

3.5 The Topological View

Recall from Theorem 3.17 that $\Sigma^{\mathbb{N}}$ is like a topology on $\mathbb{N}$. In fact, for any set A, the family of semidecidable predicates Σ^A is a σ-frame: $\emptyset$ and A are the least and the greatest elements of Σ^A, binary infima are computed as intersections, and countable suprema as unions. It makes sense then to think of Σ^A as a topology on A.

Definition 3.29 The *(intrinsic) topology* of a set A is the set Σ^A of semidecidable subsets, which are also called *open sets*. The *closed sets* are the complements of the open ones.

In the theory of effective topological spaces [17] the intrinsic topology is known as the *Eršov topology.*

Let us compare the situation with classical topology. Recall that the open subsets of a classical topological space X are in bijective correspondence with continuous maps $X \to \mathbb{S}$, where $\mathbb{S}$ is the *Sierpinski space* which consists of two points $\bot$ and $\top$, with $\{\top\}$ open and $\{\bot\}$ closed. In our setting, the correspondence holds by definition, with Σ in place of $\mathbb{S}$ and arbitrary maps in place of continuous ones. However, the maps are not as arbitrary as you might think:

Proposition 3.30 *All maps are continuous.*

Proof. For any $f : A \to B$ and an open set $U : B \to \Sigma$, the inverse image $f^*(U) = \{x \in A \mid f(x) \in U\}$ is open because its characteristic map is $U \circ f : A \to \Sigma$. Therefore f is continuous. □

We are cheating, of course, since we defined topology in such a way that all maps are trivially continuous. For a real challenge we should attempt to show, for example, that all maps $\mathbb{R} \to \mathbb{R}$ are continuous in the usual ϵ–δ sense. Although this turns out to be the case, we are *not* going to prove it here.

While in classical topology a given set may be endowed with many different topologies, in the synthetic world each set has precisely one topology associated with it. At this point you should be worried that certain sets might be equipped with the "wrong topology". For example, there are at least two important topologies on the dual of a Banach space. How can we have both if a set is only allowed to have one? The answer is that sets which are the same classically may be different constructively, and so they may carry different topologies. In the case of the dual of Banach spaces, one might consider the bounded linear functionals versus the normed linear functionals. These might turn out to be different sets, each with its own well known topology.

If we were going to develop the topological point of view futher, we would borrow ideas from *Synthetic Domain Theory* [9,15], *Abstract Stone Duality* [19], and *Synthetic Topology* [4]. We leave such an task for the future. However, we shall keep the topological point of view in mind and use its terminology whenever convenient.

4 Basic Computability Theory

In this section we introduce the Enumerability Axiom and derive the basic theorems of computability theory.

4.1 Partial Functions and Partial Values

In classical computability theory the computable partial functions are characterized as precisely those partial functions whose graph is computably enumerable. This characterization helps us find the corresponding notion in synthetic computability.

A *partial function* $f : A \rightharpoonup B$ is a function $f : A' \to B$ defined on a subset $A' \subseteq A$, called the *support* of f. Equivalently, such an f corresponds to a (total) function $g : A \to \widetilde{B}$ where $\widetilde{B}$ is the set of *partial values*

$$\widetilde{B} = \{s \in \mathcal{P}B \mid \forall\, x, y \in B \,.\, (x \in s \land y \in s \implies x = y)\} \ .$$

The connection between f and g is $g(x) = \{f(x) \in B \mid x \in A'\}$. Observe that the partial values are nothing but subsingletons. The empty set represents the value "undefined" and is denoted by $\bot_B = \emptyset$, while a singleton $\{y\}$ represents a "fully defined" value $y \in B$, which we call a *total* value. The singleton map

$\{-\} : B \to \widetilde{B}$ is an inclusion that maps the elements of B precisely onto the total values of $\widetilde{B}$. We shall often identify $y \in B$ with its representation as a total value $\{y\} \in \widetilde{B}$. The statement $\exists\, y \in B\,.\,(s = \{y\})$ means "the partial value s is total" and is abbreviated as $s\!\downarrow$.

The *graph* of $f : A \to \widetilde{B}$ is the set

$$\Gamma(f) = \big\{\langle x, y\rangle \in A \times B \;\big|\; f(x) = \{y\}\big\} \ .$$

Among all partial functions $\mathbb{N} \to \widetilde{\mathbb{N}}$ we are only interested in those that have enumerable graphs.

Proposition 4.1 *A partial function $f : \mathbb{N} \to \widetilde{\mathbb{N}}$ has an enumerable graph if, and only if, $f(n)\!\downarrow$ is semidecidable for every $n \in \mathbb{N}$.*

Proof. If $e : \mathbb{N} \to \mathbf{1} + \Gamma(f)$ is an enumeration of $\Gamma(f)$ then

$$f(n)\!\downarrow \iff (\exists\, m \in \mathbb{N}\,.\, f(n) = \{m\}) \iff (\exists\, k \in \mathbb{N}\,.\, e(k) \neq \star \wedge \pi_1\,(e(k)) = m) \in \Sigma \ .$$

Conversely, suppose $f(n)\!\downarrow \in \Sigma$ for every $n \in \mathbb{N}$. Observe that $\langle n, k\rangle \in \Gamma(f)$ if, and only if, $f(n)\!\downarrow \wedge\, f(n) = \{k\}$. If $f(n)\!\downarrow$ then there is a unique $m \in \mathbb{N}$ such that $f(n) = \{m\}$, therefore $f(n) = \{k\}$ is semidecidable, even decidable, because it is equivalent to $m = k$. By the dominance axiom, $f(n)\!\downarrow \wedge\, f(n) = \{k\}$ is semidecidable so $\Gamma(f)$ is a semidecidable subset of $\mathbb{N} \times \mathbb{N}$, hence enumerable. $\square$

We single out those partial values whose totality is semidecidable.

Definition 4.2 The *lifting* $A_\perp$ of A is the set of Σ-*partial values*,

$$A_\perp = \Big\{s \in \widetilde{A} \;\Big|\; s\!\downarrow \in \Sigma\Big\} \ .$$

A Σ-*partial function* is a partial function $f : A \to B_\perp$.

The operation $A \mapsto A_\perp$ is a functor. It acts on a map $f : A \to B$ by

$$f_\perp(p) = \big\{y \in B \;\big|\; \exists\, x \in A\,.\, x \in p \wedge f(x) = y\big\} \ .$$

This is well defined because $f_\perp(p)\!\downarrow \iff p\!\downarrow$. Thus $f_\perp(\{x\}) = \{f(x)\}$ and $f_\perp(\perp_A) = \perp_B$. For those familiar with category theory we mention that the lifting functor $-_\perp$ is in fact a monad whose multiplication and unit are union and singleton, respectively:

$$\begin{aligned}
&\mu_A : A_{\perp\perp} \to A_\perp && \eta_A : A \to A_\perp \\
&\mu_A : s \mapsto \cup s = \big\{x \in A \;\big|\; \exists\, p \in A_\perp\,.\, p \in S \wedge x \in p\big\} && \eta_A : x \mapsto \{x\}
\end{aligned}$$

The Σ-partial functions $\mathbb{N} \to \mathbb{N}_\perp$ are the synthetic analogue of partial computable functions. A classical theorem of computability theory claims that the computably enumerable sets are precisely the supports of partial computable functions.

Proposition 4.3

(i) *A partial function is Σ-partial if, and only if, its support is semidecidable.*

(ii) *A subset is semidecidable if, and only if, it is the support of a Σ-partial function.*

Proof.

(i) The support of $f : A \to \widetilde{B}$ is the set $\{x \in A \mid f(x){\downarrow}\}$. Clearly then, the support is semidecidable if, and only if, totality is semidecidable.

(ii) We already proved that the support of a Σ-partial function is semidecidable. Conversely, if $S \subseteq A$ is semidecidable then it is the support of its characteristic function $\chi_S : A \to \Sigma = \mathbf{1}_\perp$.

$\square$

Among all semidecidable subsets of $\mathbb{N} \times \mathbb{N}$ we may identify those that are graphs of Σ-partial functions. Recall the definition of single-valued relation from 2.2.

Proposition 4.4 *The graphs of Σ-partial functions $\mathbb{N} \to \mathbb{N}_\perp$ are precisely all the single-valued semidecidable relations on $\mathbb{N} \times \mathbb{N}$.*

Proof. The graph of a Σ-partial function is semidecidable because it is enumerable, and it is clearly single valued. Given a single-valued $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$, define $f_R : \mathbb{N} \to \mathbb{N}_\perp$ by $f_R(m) = \{n \in \mathbb{N} \mid \langle m, n\rangle \in R\}$. Then $\Gamma(f_R) = R$. $\square$

A *selection* for a binary relation $R \subseteq A \times B$ is a *partial* function $s : A \to \widetilde{B}$ such that, for all $x \in A$,

$$(\exists\, y \in B \,.\, R(x, y)) \implies s(x){\downarrow} \wedge R(x, s(x)) \;.$$

The selection function s is like a choice function for R, except that it is defined only at those arguments for which there is something to choose from. A well known theorem in computability theory, the Single Value Theorem, says that every semidecidable relation on $\mathbb{N} \times \mathbb{N}$ has a Σ-partial selection.

Theorem 4.5 (Single Value Theorem) *Every semidecidable binary relation on $\mathbb{N}$ has a Σ-partial selection.*

Proof. Let $e : \mathbb{N} \to \mathbf{1} + R$ be an enumeration of $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$. Define S by

$$\begin{aligned} S = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid \\ \exists\, k \in \mathbb{N} \,.\, (e(k) = \langle m, n\rangle \wedge \forall\, j < k \,.\, (e(j) \neq \star \implies \pi_1\,(e(j)) \neq m))\} \;. \end{aligned}$$

Thus we put $\langle m, n\rangle$ in S when it is the first pair of the form $\langle m, -\rangle$ enumerated by e. Clearly $S \subseteq R$. To see that it is single-valued, suppose $\langle m, n\rangle \in S$ and

$\langle m, n' \rangle \in S$. Then $\langle m, n \rangle = e(k)$ and $\langle m, n' \rangle = e(k')$ for some $k, k' \in \mathbb{N}$. It is impossible that $k' < k$ or $k < k'$, therefore $k = k'$ and so $n = n'$. Lastly, if $\langle m, n \rangle \in R$ then $\langle m, n \rangle = e(k)$ for some $k \in \mathbb{N}$. Now there is a least $j \leq k$ such that $\pi_1(e(j)) = m$. Then $e(j) = \langle m, n' \rangle \in S$. A selection for R is the function whose graph is S. $\square$

4.2 The Enumerability Axiom

Everything we have considered so far is consistent with classical logic. Of course, if we interpreted all the definitions and theorems in classical set theory, we would not discover anything interesting, as it would turn out that $2 = \Sigma = \Omega$, all predicates and sets are decidable, all spaces are compact and overt, etc. It is time to introduce a genuinely interesting axiom.

Axiom 4.6 (Enumerability) *There are enumerably many enumerable subsets of* $\mathbb{N}$.

Let $\mathsf{W}_{-} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be such an enumeration.

The idea for the Enumerability Axiom comes from the Enumeration Theorem of classical computability theory, which states that there is a computable enumeration of computably enumerable sets. In the classical theory there is also an enumeration theorem for partial computable functions. We have it as well.

Proposition 4.7 $\mathbb{N} \to \mathbb{N}_\perp$ *is enumerable.*

Proof. By Enumerability Axiom there is an enumeration $V_{-} : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$, because $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$. By Single Value Theorem, for every $n \in \mathbb{N}$ there exists a selection for V_n. By Number Choice there is a choice function $\varphi_{-} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\perp)$ such that φ_n is a selection for V_n. We are done because φ is surjective: for any $f : \mathbb{N} \to \mathbb{N}_\perp$ there is $n \in \mathbb{N}$ such that $V_n = \Gamma(f)$, but then $\varphi_n = f$ because f is the only selection for $\Gamma(f)$. $\square$

Let $\varphi_{-} : \mathbb{N} \twoheadrightarrow (\mathbb{N} \to \mathbb{N}_\perp)$ be an enumeration. Next we derive some basic consequences of the Enumeration Axiom.

Proposition 4.8 Σ *and* $\mathcal{E}$ *have the fixed-point property.*

Proof. By Proposition 3.13 together with the observation that $\mathcal{E}^{\mathbb{N}} = (\Sigma^{\mathbb{N}})^{\mathbb{N}} \cong \Sigma^{\mathbb{N}\times\mathbb{N}} \cong \Sigma^{\mathbb{N}} = \mathcal{E}$. $\square$

Corollary 4.9 *None of the inclusions* $2 \subseteq \Sigma \subseteq \Omega_{\neg\neg} \subseteq \Omega$ *is an equality.*

Proof. Neither 2 nor $\Omega_{\neg\neg}$ has the fixed-point property so they cannot be equal to Σ. If $\Omega_{\neg\neg} = \Omega$ then $2 = \Omega$, see Exercise 2.5, but we already have $2 \neq \Sigma \subseteq \Omega$, hence $\Omega_{\neg\neg} \neq \Omega$. $\square$

The Enumerability Axiom invalidates classical logic because it falsifies the Law of Excluded Middle, $2 = \Omega$, which contradicts Corollary 4.9.

Because $2 \neq \Sigma$ the decidable and the semidecidable subsets of $\mathbb{N}$ are not the same. We may explicitly construct a semidecidable subset which is not decidable, namely the well known

$$\mathsf{K} = \left\{ n \in \mathbb{N} \,\middle|\, n \in \mathsf{W}_n \right\} \ .$$

The set K is not decidable because its complement $\mathbb{N} \setminus \mathsf{K}$ is not semidecidable. If it were there would be some $m \in \mathbb{N}$ such that $\mathsf{W}_m = \mathbb{N} \setminus \mathsf{K}$ and then we would have the usual contradiction

$$m \in \mathsf{K} \iff m \in \mathsf{W}_m \iff m \in \mathbb{N} \setminus \mathsf{K} \iff m \notin \mathsf{K} \ .$$

Recall that Σ is a σ-frame. Its partial order $p \leq q$ is logical implication $p \implies q$. It is important to know how maps $\Sigma \to \Sigma$ interact with the partial order.

Proposition 4.10 *Every map $f : \Sigma \to \Sigma$ is monotone.*

Proof. The statement of monotonicity of f is classical:

$$\forall\, p, q \in \Sigma \,.\, (p \leq q \implies f(p) \leq f(q)) \ .$$

By Lemma 3.27 this statement reduces to checking all four combinations of $p, q \in \{\bot, \top\}$. Of these only $p = \bot$, $q = \top$ is nontrivial so that monotonicity of f reduces to

$$f(\bot) \leq f(\top) \ .$$

Suppose the opposite of this, which is $f(\bot) \wedge \neg f(\top)$, were true. Then $f(\bot) = \top$ and $f(\top) = \bot$. Now $p = f(p)$ implies $p \neq \bot$ and $p \neq \top$, which is impossible. Hence f does not have a fixed point, which cannot be because Σ has the fixed-point property. We proved $\neg\neg(f(\bot) \leq f(\top))$, hence $f(\bot) \leq f(\top)$ as desired. □

Proposition 4.11 (Phoa's Principle) *For every $f : \Sigma \to \Sigma$,*

$$f(x) = (f(\bot) \vee x) \wedge f(\top) \qquad \textit{and} \qquad f(x) = f(\bot) \vee (x \wedge f(\top)) \ .$$

Proof. Phoa's principle is a classical statement so we only need to check $x = \bot$ and $x = \top$. This gives us four equations of which two are trivially true and the other two are

$$f(\bot) = f(\bot) \wedge f(\top) \qquad \text{and} \qquad f(\top) = f(\bot) \vee f(\top) \ .$$

Both of these are equivalent to $f(\bot) \leq f(\top)$, which holds because f is monotone. □

The relevance of Phoa's Principle is revealed by the following corollary.

Corollary 4.12 *Every map $f : \Sigma \to \Sigma$ preserves binary infima and countable suprema.*

Proof. By Phoa's Principle,

$$f(p \wedge q) = f(\bot) \vee (p \wedge q \wedge f(\top)) = f(\bot) \vee ((p \wedge f(\top)) \wedge (q \wedge f(\top))) = \\ (f(\bot) \vee (p \wedge f(\top))) \wedge (f(\bot) \vee (p \wedge f(\top))) = f(p) \wedge f(q) \ ,$$

and

$$f(\textstyle\bigvee_n p_n) = (f(\bot) \vee \bigvee_n p_n) \wedge f(\top) = (\bigvee_n f(\bot) \vee p_n) \wedge f(\top) = \\ \textstyle\bigvee_n (f(\bot) \vee p_n) \wedge f(\top) = \bigvee_n f(p_n) \ .$$

□

Proposition 4.13 *If A is classical then so is $A_\bot$.*

Proof. Suppose $s, t \in A_\bot$ and $\neg\neg(s = t)$. Consider any $x \in A$. If $x \in s$ then $\neg\neg(x \in t)$ and $x \in t$ because $(x \in t) \in \Sigma \subseteq \Omega_{\neg\neg}$. This shows $s \subseteq t$ and a symmetric argument establishes $t \subseteq s$. □

4.3 Inseparable Sets

A *separation* of subsets $A_0, A_1 \subseteq \mathbb{N}$ is a decidable subset $D \in 2^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $n \in A_0 \implies n \notin D$ and $n \in A_1 \implies n \in D$. Clearly, if two sets can be separated they are disjoint.

A pair of disjoint subsets $A_0, A_1 \subseteq \mathbb{N}$ is *weakly inseparable* there is no separation for them. It is *inseparable* if, for every $D \in 2^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $n \in A_0$ and $n \in D$, or $n \in A_1$ and $n \notin D$.

Proposition 4.14 *There exist a pair of open inseparable subsets of $\mathbb{N}$.*

Proof. The standard proof works. We claim that $A_0 = \{n \in \mathbb{N} \mid \varphi_n(n) = 0\}$ and $A_1 = \{n \in \mathbb{N} \mid \varphi_n(n) = 1\}$ are such sets. Consider any $D \in 2^{\mathbb{N}}$ and define a map $d : \mathbb{N} \to \mathbb{N}$ by $d(k) = \texttt{if}\ k \in D\ \texttt{then}\ 0\ \texttt{else}\ 1$. There exists $n \in \mathbb{N}$ such that $d = \varphi_n$. If $d(n) = 0$ then $n \in A_0$ and $n \in D$, whereas if $d(n) = 1$ then $n \in A_1$ and $n \notin D$. □

An interesting consequence of the existence of inseparable open sets is that the Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are isomorphic, which we prove next.

Recall that a *binary tree* T is a prefix-closed subset of $\mathsf{Seq}\,2$, where $\mathsf{Seq}\,2$ is the set of finite binary sequences. We call the elements of T *branches.* A *leaf* is a branch that cannot be extended any further. A *Kleene tree* is a binary tree K such that:

(i) K is an inhabited decidable subset of $\mathsf{Seq}\,2$,

(ii) K contains arbitrarily tall branches: for every $n \in \mathbb{N}$ there is $a \in K$ whose length is at least n,

(iii) every infinite path exists K: for every $\alpha \in 2^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $\alpha_0\alpha_1 \dots \alpha_n \notin K$.

Proposition 4.15 *There exists a Kleene tree.*

Proof. Again, the standard proof works. Let $e : \mathbb{N} \to A_0$ and $f : \mathbb{N} \to A_1$ be enumerations of a pair of open inseparable sets. For $n \in \mathbb{N}$, let $A_0^n = \{e(m) \mid m < n\}$ and $A_1^n = \{f(m) \mid m < n\}$. For a binary sequence $a = a_0 \dots a_n$ define

$$a \in T \iff \forall m \leq n \,.\, ((a_m = 0 \wedge m \in A_0^n) \vee (a_m = 1 \wedge m \in A_1^n)) \ .$$

This defines a subset $T \subseteq \mathsf{Seq}\, 2$ which is decidable and inhabited by the empty sequence. Because both A_0 and A_1 contain infinite sequences, T contains arbitrarily long branches. But T does not contain any infinite paths. For any $\alpha \in 2^{\mathbb{N}}$ there exists $k \in \mathbb{N}$ such that $\alpha_k = 0 \wedge k \in A_1$ or $\alpha_k = 1 \wedge k \in A_0$. There exists $n \in \mathbb{N}$ such that $e(n) = k$ or $f(n) = k$. In either case, $\alpha_0 \dots \alpha_{n+1} \notin K$.□

The set L of leaves of Kleene tree K is decidable and contains an infinite sequence. By Proposition 3.10 there is an isomorphism $i : \mathbb{N} \to L$. Now we can define an isomorphism $h : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ as follows. For $\beta \in \mathbb{N}^{\mathbb{N}}$, let $h(\beta)$ be the sequence $i(\beta_0)i(\beta_1)i(\beta_2)\cdots$. The inverse h^{-1} exists, too. Given a binary sequence $\alpha \in 2^{\mathbb{N}}$, find that prefix $\alpha_0 \dots \alpha_n$ which is a leaf in T. Let $\beta_0 = i^{-1}(\alpha_0 \dots \alpha_n)$. Now "chop off" the prefix $\alpha_0 \dots \alpha_n$ and repeat to define $\beta_1, \beta_2, \dots$ This gives us a sequence $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $h(\beta) = \alpha$. This completes the sketch of a proof of the following proposition.

Proposition 4.16 *Cantor space $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are isomorphic.*

4.4 Focal Sets

The lifting operation attaches to a set A an "undefined" value $\bot_A$. Sometimes a set already contains an element which plays the role of "undefined"; for example, in the set of Σ-partial functions $A \to B_\bot$ it is the everywhere undefined function $\lambda x : A \,.\, \bot_B$. Such a special element can be found by attaching $\bot$ to the set and mapping it back in to the original set, without changing the original elements. This idea leads to the following definition.

Definition 4.17 A *focal set* is a set A together with a mapping $\epsilon : A_\bot \to A$, called the *focus map*, such that $\epsilon(\{x\}) = x$ for all $x \in A$. The element $\epsilon(\bot_A)$ is called the *focus point*.

We usually denote the focus point by $\bot$.

A lifted set $A_\bot$ is focal. The focus map is the multiplication μ_A for the lifting monad, as defined in the paragraph following Definition 4.2. The focus point is $\bot_A$, as expected.

If B is a focal set with focus map $\delta : B_\bot \to B$ then $A \to B$ is a focal set with focus map $\epsilon : (A \to B)_\bot \to (A \to B)$ defined by

$$\epsilon(p) = \lambda x : A \,.\, \delta(\{y \in B \mid \exists f : A \to B_\bot \,.\, (f \in p \wedge f(x) = \{y\})\}) \ . \qquad (2)$$

The focus point of $A \to B$ is the map that maps every element to the focus point of B.

By combining the last two observations, we see that the set of Σ-partial maps $A \to B_\perp$ is focal.

A product of focal sets A and B is focal. The focus map $\epsilon : (A \times B)_\perp \to A \times B$ is $\epsilon(p) = \langle \epsilon_A(\pi_{1\perp}(p)), \epsilon_B(\pi_{2\perp}(p)) \rangle$, and the focus point is the pair whose components are the foci of A and B.

A set can have two focal structures, for example, the maps $\delta : \Omega_\perp \to \Omega$ and $\epsilon : \Omega_\perp \to \Omega$ defined by

$$\delta(s) = \exists\, p \in s \,.\, p \qquad \text{and} \qquad \epsilon(s) = \forall\, p \in s \,.\, p$$

both satisfy $\delta(\{q\}) = q$ and $\epsilon(\{q\}) = q$, so they define focal structures on Ω, but $\delta(\emptyset) = \bot$ and $\epsilon(\emptyset) = \top$.

The enumerable focal sets are know in the theory of numbered sets as *Eršov complete sets* [3]. They have good properties.

Lemma 4.18 *(a) If $f : A \to B$ is onto then so is $f_\perp : A_\perp \to B_\perp$. (b) If $f : A \to B$ is onto then so is $f^{\mathbb{N}} : A^{\mathbb{N}} \to B^{\mathbb{N}}$.*

Proof. (a) For any $s \in B_\perp$ we have $f_\perp(f^{-1}(s)) = s$ and since $s{\downarrow} \iff f^{-1}(s){\downarrow}$, $f^{-1}(s) \in A_\perp$.

(b) The map $f^{\mathbb{N}}$ is defined by $f^{\mathbb{N}}(g) = f \circ g$. Consider any $h : \mathbb{N} \to B$. For every $n \in \mathbb{N}$ there is $x \in A$ such that $f(x) = h(n)$. By Number Choice there is $g : \mathbb{N} \to A$ such that $f(g(n)) = h(n)$ for all $n \in \mathbb{N}$. Hence $f^{\mathbb{N}}(g) = h$. $\square$

Proposition 4.19 *If A is an enumerable focal set then so is $A^{\mathbb{N}}$.*

Proof. Let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration and $\epsilon : A_\perp \to A$ the focal map. By the previous lemma, we have a chain of epimorphisms

$$\mathbb{N} \xrightarrow{\ \varphi\ } \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{\ e_\perp^{\mathbb{N}}\ } A_\perp^{\mathbb{N}} \xrightarrow{\ \epsilon^{\mathbb{N}}\ } A^{\mathbb{N}} \ .$$

This proves that $A^{\mathbb{N}}$ is enumerable. It is focal by (2). $\square$

4.5 Rice's Theorem

A set A is *connected* if it cannot be decomposed as a disjoint union $A_1 + A_2$ in a non-trivial way. An equivalent way of saying this is that every map $A \to 2$ is constant, which we use as the definition of connectedness.

Definition 4.20 A set A is *connected* if every map $A \to 2$ is constant.

Proposition 4.21 *Σ is connected.*

Proof. Consider a map $h : \Sigma \to 2$. Let $r : 2 \to \Sigma$ be the map

$$r(p) = \texttt{if } p = h(\bot) \texttt{ then } \top \texttt{ else } \bot \,.$$

By Proposition 4.10 the map $r \circ h$ is monotone so that for every $x \in \Sigma$ we have

$$\top = r(h(\bot)) \leq r(h(x)) \leq \top \ .$$

Thus $r(h(x)) = \top$, hence $h(x) = h(\bot)$ by definition of r. $\square$

Lemma 4.22 *Let A be a focal set with focus $\bot_A$. For every $x \in A$ there exists $f : \Sigma \to A$ such that $f(\bot) = \bot_A$ and $f(\top) = x$.*

Proof. Let $g : \mathbf{1} \to A$ be the map $g(\star) = x$. Define $f(s) = \epsilon(g_\bot(s))$ where $\epsilon : A_\bot \to A$ is the focal map. Then $f(\bot) = \epsilon(g_\bot(\bot)) = \epsilon(\bot) = \bot_A$ and $f(\top) = \epsilon(g_\bot(\top)) = \epsilon(x) = x$, as required. $\square$

Theorem 4.23 (Rice's Theorem) *A focal set is connected.*

Proof. Let $h : A \to 2$ be an arbitrary map. We show that $h(x) = h(\bot_A)$ for every $x \in A$. As in Lemma 4.22, let $f : \Sigma \to A$ be such that $f(\bot) = \bot_A$ and $f(\top) = x$. Because Σ is connected $h \circ f$ is constant, but this means $h(x) = h(f(\top)) = h(f(\bot)) = h(\bot_A)$. $\square$

The classical Rice's Theorem states that there are no non-trivial decidable subsets of $\mathcal{E}$. This follows immediately from our theorem as $\mathcal{E}$ is focal.

4.6 Recursion Theorem

A *multivalued function* $f : A \rightrightarrows B$ is a function $f : A \to \mathcal{P}B$ such that $f(x)$ is inhabited for every $x \in A$. The graph of a multivalued function $\Gamma(f) \subseteq A \times B$, defined by

$$\Gamma(f) = \left\{ \langle x, y \rangle \in A \times B \;\middle|\; y \in f(x) \right\} \ ,$$

is a total relation. Every total relation $R \subseteq A \times B$ determines a multivalued function $f_R : A \rightrightarrows B$ by $f_R(x) = \left\{ y \in B \;\middle|\; R(x, y) \right\}$, hence multivalued functions and total relations are two equivalent notions.

A *fixed point* of a multivalued function $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Theorem 4.24 (Recursion Theorem) *Every multivalued function on an enumerable focal set has a fixed point.*

Proof. Let $f : A \rightrightarrows A$ be a multivalued function, let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration, and $\epsilon : A_\bot \to A$ a focal map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $e(m) \in f(e(k))$. By Number Choice there is a map $c : \mathbb{N} \to \mathbb{N}$ such that $e(c(k)) \in f(e(k))$ for every $k \in \mathbb{N}$. It suffices to find k such that $e(c(k)) = e(k)$ since then we can take $x = e(k)$.

For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\epsilon(e_\bot(c_\bot(\varphi_m(m)))) = e(n)$. By Number Choice there is $g : \mathbb{N} \to \mathbb{N}$ such that $\epsilon(e_\bot(c_\bot(\varphi_m(m)))) = e(g(m))$ for every $m \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $g = \varphi_j$. Let $k = g(j)$. Then

$$e(k) = e(g(j)) = \epsilon(e_\bot(c_\bot(\varphi_j(j)))) = \epsilon(e_\bot(c_\bot(g(j)))) = e(c(g(j))) = e(c(m)) \ .$$

□

The classical Recursion Theorem is indeed a consequence of what we just proved.

Corollary 4.25 *For every $f : \mathbb{N} \to \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\varphi_{f(n)} = \varphi_n$.*

Proof. Apply Recursion Theorem to the enumerable focal set $\mathbb{N} \to \mathbb{N}_\perp$ and the multivalued function h, defined by

$$h(u) = \left\{ v : \mathbb{N} \to \mathbb{N}_\perp \;\middle|\; \exists\, n \in \mathbb{N} \,.\, u = \varphi_n \wedge v = \varphi_{f(n)} \right\} ,$$

to obtain a fixed point $u \in h(u)$. By definition of h there exists $n \in \mathbb{N}$ such that $u = \varphi_n$ and $u = \varphi_{f(n)}$, hence $\varphi_n = \varphi_{f(n)}$. □

The following consequence of Recursion Theorem is a generalization of Berger's Branching Lemma [1] and will be useful in characterizing the open subsets of ω-chain complete posets.

Lemma 4.26 (Berger) *Let A be an enumerable focal set, $U : A \rightrightarrows \Sigma$ a multivalued open set, and $x : \mathbb{N}^+ \to A$ a sequence with a limit. If $U(x_\infty) = \{\top\}$ then $\top \in U(x_n)$ for some $n \in \mathbb{N}$.*

Proof. Recall that by Proposition 3.23 for every $s \in \Sigma$ there exists $p \in \mathbb{N}^+$ such that $s = \top \iff p < \infty$. Therefore, for every $y \in A$ there is $p \in \mathbb{N}^+$ such that $(p < \infty) \in U(y)$. Consequently, for every $y \in A$ there is $z \in A$ such that

$$\exists\, p \in \mathbb{N}^+ \,.\, ((p < \infty) \in U(y) \wedge z = x_p) \; . \tag{3}$$

By Recursion Theorem there is $y = z$ satisfying (3). For such y, p is not equal to ∞ because $p = \infty$ implies $y = x_\infty$ and $\perp = (p < \infty) \in U(y) = U(x_\infty) = \{\top\}$, contradiction. By Exercise 3.4, $p \in \mathbb{N}$ so we have $x_p = y$ and $\top = (p < \infty) \in U(x_p)$, as required. □

4.7 The Myhill-Shepherdson and Rice-Shapiro Theorems

Recall that a poset $(A, \leq)$ is *ω-chain complete* if every increasing chain $x_0 \leq x_1 \leq x_2 \leq \cdots$ has a supremum $\bigvee_n x_n$. A subset $S \subseteq A$ *generates* A if every element in A is the supremum of a chain in S. A *base* for A is an enumerable subset $S \subseteq A$ that generates A, and $x \leq y$ is semidecidable whenever $x \in S$ and $y \in A$.

Proposition 4.27 *The following sets are ω-chain complete with a base, therefore countably based:*

(i) *Σ with the base $\{\top\}$,*

(ii) *$\mathbb{N}_\perp$ with the base $\mathbb{N}$,*

(iii) *$A^{\mathbb{N}}$, if A is an ω-chain complete focal set with a base.*

Proof.

(i) For any $x \in \Sigma$ we have $x = \bigvee \{\top \mid x\}$.

(ii) For any $x \in \mathbb{N}_\perp$ we have $x = \{n \in \mathbb{N} \mid n \in x\} = \bigvee \{n \in \mathbb{N} \mid n \in x\}$.

(iii) The partial order on $A^{\mathbb{N}}$ is component-wise: $f \leq g \iff \forall n \in \mathbb{N} \,.\, f(n) \leq g(n)$. This makes $A^{\mathbb{N}}$ into an ω-chain complete poset with the supremum of a chain computed component-wise. If S is a base for A, we may take as a base for $A^{\mathbb{N}}$ the set

$$T = \{f \in \mathbb{A}^{\mathbb{N}} \mid \exists n \in \mathbb{N} \,.\, \forall k \geq n \,.\, f(k) = \perp_A\} \ .$$

It is not hard to see that T is enumerable. Given any $f \in A^{\mathbb{N}}$, there is by Countable Choice a map $g : \mathbb{N} \to S^{\mathbb{N}}$ such that, for every $m \in \mathbb{N}$, $g(m)$ is a chain in S with supremum $f(m)$. Define $h : \mathbb{N} \to T$ by

$$h(n)(m) = \texttt{if } m < n \texttt{ then } g(m)(n) \texttt{ else } \perp_A \ .$$

Then h is a chain in T whose supremum is f:

$$\bigvee_n h(n)(m) = \bigvee_{n>m} h(n)(m) = \bigvee_{n>m} g(m)(n) = f(m) \ .$$

□

By the previous proposition $\Sigma^{\mathbb{N}}$ and $\mathbb{N} \to \mathbb{N}_\perp$ are ω-chain complete with bases consisting of the finite subsets of $\mathbb{N}$ and the finite Σ-partial maps, respectively. A map $f : \mathbb{N} \to \mathbb{N}_\perp$ is *finite* if it has finite support.

Theorem 4.28 *In an ω-chain complete poset $(A, \leq)$ open subsets are upward closed and inaccessible by chains, i.e., if the supremum of a chain belongs to an open set then already some element of the chain does.*

Furthermore, if $S \subseteq A$ is a base for A, then every open subset of A is an enumerable union of open subsets of the form $\uparrow x = \{y \in A \mid x \leq y\}$ with $x \in S$.

Proof. For the first claim, suppose $x \in U \in \Sigma^A$ and $x \leq y$. Define a sequence $a : \mathbb{N}^+ \to A$ by

$$a_p = \bigvee_{n \in \mathbb{N}} (\texttt{if } n < p \texttt{ then } x \texttt{ else } y)$$

Then $a_\infty = x \in U$ and by Berger's Lemma there exists $k < \infty$ such that $y = a_k \in U$. For the second claim, suppose $x_0 \leq x_1 \leq \cdots$ is a chain with $\bigvee_n x_n \in U \in \Sigma^A$. Define $b : \mathbb{N}^+ \to A$ by

$$b_p = \bigvee_{n \in \mathbb{N}} x_{\min(n,p)} \ .$$

Then $b_\infty = \bigvee_n x_n \in U$ and once again by Berger's Lemma there exists $k < \infty$ such that $x_k = b_k \in U$.

For the last claim, suppose U is open. Then $T = S \cap U$ is enumerable. By the first claim $\bigcup_{x \in T} \uparrow x \subseteq U$, and the opposite inclusion holds as well: if

$y \in U$, then $y = \bigvee_n x_n$ for some chain $x_0 \leq x_1 \leq \cdots$ in S, therefore by the second claim $x_k \in U$ for some $k \in \mathbb{N}$. But then $y \in \uparrow x_k \subseteq \bigcup_{x \in T} \uparrow x$, as required. $\square$

The Myhill-Shepherdson and Rice-Shapiro theorems characterize the topologies of $\mathbb{N} \to \mathbb{N}_\perp$ and $\Sigma^{\mathbb{N}}$, respectively. They follow immediately from the previous proposition and theorem.

Corollary 4.29 (Myhill-Shepherdson) *A subset U of $\mathbb{N} \to \mathbb{N}_\perp$ is open if, and only if, it is an enumerable union $U = \bigcup_{n \in \mathbb{N}} \uparrow f_n$, where each $f_n : \mathbb{N} \to \mathbb{N}_\perp$ has finite support.*

Corollary 4.30 (Rice-Shapiro) *A subset $U \subseteq \Sigma^{\mathbb{N}}$ is open if, and only if, it is an enumerable union $U = \bigcup_{n \in \mathbb{N}} \uparrow S_n$, where each $S_n \subseteq \mathbb{N}$ is finite.*

5 Conclusion

We have only scratched the surface of a large body of work. There are at least two directions to go from here.

First, we could develop recursive topology [17] and recursive analysis in a style that resembles the usual topology and analysis, but with unusual results, such as failure of compactness of the closed interval and the existence of open subsets of Cantor space that are not metrically open. But we could also prove positive results, such as the Kreisel-Lacombe-Shoenfield theorem, which states that all functions between complete separable metric spaces are continuous in the metric sense.

Second, we have not spoken at all about Turing reducibility and Turing degrees. How this can be done with the j-operators in the effective topos was indicated by Hyland [8]. There might be simpler ways to treat Turing degrees. In particular, it is well known that the priority methods are related to Baire category theorem [13, V.3], a connection worth examining in the synthetic setting.

The axiomatization presented in this paper has its limit: it cannot prove any results in computability theory that fail to relativize to oracle computations. This is so because the theory can be interpreted in a variant of the effective topos built from partial recursive functions with access to an oracle.

References

[1] Berger, U., *Total sets and objects in domain theory*, Annals of Pure and Applied Logic **60** (1993), pp. 91–117.

[2] Davis, M., “Computability and Unsolvability,” McGraw-Hill, 1958, reprinted in 1982 by Dover Publications.

[3] Ershov, Y., “The theory of enumerations,” Nauka, 1980.

[4] Escardó, M., *Synthetic topology of data types and classical spaces*, Electronic Notes in Theoretical Computer Science **87** (2004), pp. 21–156.

[5] Fenstad, J., *On axiomatizing recursion theory*, in: J.E. Fenstadt et al., editor, *Generalized Recursion Theory*, North Holland, 1974 pp. 385–404.

[6] Friedman, H., *Axiomatic recursive function theory*, in: Gandy et al., editor, *Logic Colloquium '69* (1971), pp. 113–137.

[7] H.R. Rogers, J., "Theory of Recursive Functions and Effective Computability," MIT Press, 1992, 3rd edition.

[8] Hyland, J., *The effective topos*, in: A. Troelstra and D. V. Dalen, editors, *The L.E.J. Brouwer Centenary Symposium* (1982), pp. 165–216.

[9] Hyland, J., *First steps in syntehtic domain theory*, in: *Category Theory*, number 1488 in Lecture Notes in Mathematics, 1991.

[10] McCarty, D., "Realizability and Recursive Mathematics," D.Phil. Thesis, University of Oxford (1984).

[11] Moschovakis, Y., *Axioms for computation theories – first draft*, in: Gandy et al., editor, *Logic Colloquium '69* (1971), pp. 199–255.

[12] Mulry, P., *Generalized Banach-Mazur functionals in the topos of recursive sets*, Journal of Pure and Applied Algebra **26** (1982), pp. 71–83.

[13] Odifreddi, P., "Classical Recursion Theory," Studies in logic and the foundations of mathematics **125**, North-Holland, 1989.

[14] Richman, F., *Church's thesis without tears*, The Journal of Symbolic Logic **48** (1983), pp. 797–803.

[15] Rosolini, G., "Continuity and Effectiveness in Topoi," Ph.D. thesis, University of Oxford (1986).

[16] Soare, R., "Recursively Enumerable Sets and Degrees," Perspectives in Mathematical Logic, Springer Verlag, 1987.

[17] Spreen, D., *On effective topological spaces*, The Journal of Symbolic Logic **63** (1998), pp. 185–221.

[18] Taylor, P., "Practical Foundations of Mathematics," Number 59 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

[19] Taylor, P., *Geometric and higher order logic using abstract Stone duality*, Theory and Applications of Categories **15** (2000), pp. 284–338.

[20] Troelstra, A. and D. v. Dalen, "Constructivism in Mathematics, Volume 1," Studies in Logic and the Foundations of Mathematics **121**, North-Holland, Amsterdam, 1988.