Derivative Portfolio Hedging Based on CVaR*

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Abstract. The use of derivatives can lead to higher yields and lower funding costs. In addition, derivatives are indispensable tools for risk management. We analyze the derivative portfolio hedging problems based on *value at risk* (VaR) and *conditional value at risk* (CVaR). We show that these derivative portfolio optimization problems are often ill-posed and the resulting optimal portfolios frequently incur large transaction and management costs. In addition, the optimal portfolio may perform poorly under a slight model error. A CVaR optimization model including a proportional cost is proposed to produce optimal portfolios with fewer instruments and smaller transaction cost with similar expected returns and a small compromise in risk. In addition, we illustrate the importance of sensitivity testing of the hedging performance with respect to model error; the optimal portfolio under a suitable cost consideration performs much more robustly with respect to model error. Finally, we discuss computational issues for large scale CVaR optimization problems and consider a smoothing technique which solves a CVaR optimization problem more efficiently than the standard linear programming methods.

Keywords. *portfolio selection, portfolio hedging, VaR, CVaR, risk minimization, Black-Scholes model, ill-posedness, transaction and management cost*

JEL Classification. *C61 and G11*

1 Introduction

Derivative contracts are widely used by institutions and investors to achieve higher returns and decrease funding costs. In addition, the use of derivatives has fundamentally changed financial risk management by providing new tools to manage risk [10]. When derivatives are used for investment or risk management, an appropriate risk measure is chosen to evaluate the performance of a portfolio. The classical Markowitz [13] risk measure based on standard deviation is no longer appropriate since the return distributions of derivatives are typically

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not normal. A more appropriate risk measure for a portfolio of derivatives is value at risk (VaR) or conditional value at risk (CVaR).

For a given time horizon $\bar{t} > 0$ and confidence level β, the *value at risk* of a portfolio is the loss in the portfolio's market value over the time horizon $\bar{t}$ that is exceeded with probability $1 - \beta$. The Bank of International Settlements has set the confidence level β to 99% and the time horizon $\bar{t}$ to 10 days while other financial institutions may disclose their VaR at other levels, e.g., 95% or 99% level, and other time horizons. VaR has become a popular risk measure for risk management, both for the purposes of reporting and measurement of capital adequacy. Despite its wide acceptance, it has been noted that VaR is not a coherent risk measure; Artzner et al [4] define a coherent risk measure as one that satisfies the axioms of translation invariance, subaddivity, positive homogeneity, and monotonicity. VaR lacks subadditivity and convexity [4, 5] (The VaR of the combination of two portfolios can be greater than the sum of the VaR of the individual portfolios). Indeed, VaR is a coherent risk measure only when it is based on the standard deviation of normal distributions. In addition, the lack of convexity limits its use as a risk measure in selecting an optimal portfolio for investment and risk management purposes. It has been illustrated in [14] and [15] that the problem of minimizing VaR of a portfolio of derivative contracts can have multiple local minimizers.

An alternative risk measure to VaR is *conditional value at risk* (CVaR), which is also known as *mean shortfall[14]*, *expected shortfall[2]*, and *tail VaR[5]*. In the context of a continuous distribution (which we assume for simplicity in this paper), for a given time horizon $\bar{t}$ and confidence level β, CVaR is the conditional expectation of the loss above VaR for the time horizon $\bar{t}$ and the confidence level β. In contrast to VaR, CVaR provides additional information on the magnitude of the excess loss. It has been shown [16] that CVaR is a coherent risk measure. In addition, minimizing CVaR typically leads to a portfolio with a small VaR.

In this paper, we analyze the VaR and CVaR optimization problems for which the instrument universe consists of various derivative contracts. In particular, we focus on the portfolio hedging problem which uses the available instruments to decrease the risk of an existing portfolio. We illustrate that, when the decision universe is a collection of derivative contracts, the optimization problem based on either the VaR or CVaR measure is typically ill-posed. The resulting optimal portfolio usually incurs a large transaction cost as well as management cost. Moreover, it tends to magnify the model error due to extreme positions. In [3], we present similar analysis and computational results for the portfolio selection problem which minimizes risk while at the same time achieving a specified expected return.

The ill-posedness of the derivative portfolio optimization problem suggests that insufficient information is specified to obtain a desirable portfolio. Natural additional criteria to include in the portfolio decision are transaction and management costs. Management cost is difficult to model but it is reasonable to expect a higher administration cost for a portfolio with more instruments. By including a cost function which is proportional to the magnitude of the instrument holdings in a CVaR risk minimization problem, we illustrate that a hedging portfolio with a smaller transaction cost and fewer instruments can be found with an often negligible compromise in risk. Moreover, the optimal hedging portfolio obtained under a suitably large cost parameter is more robust with respect to model error.

A convex (under suitable assumptions) optimization problem has been proposed in [17] to compute the optimal CVaR portfolio when the loss of a portfolio has a continuous distribution. In addition, when this optimization problem is approximated by Monte Carlo sampling, it has an equivalent linear programming formulation and can be solved using standard linear programming methods. Unfortunately, a portfolio CVaR optimization problem involving derivative contracts can quickly become a large-scale optimization problem due to the wide variety of available derivatives and the many different specifications for each type of derivative. We illustrate that solving a CVaR optimization problem using standard linear programming software can quickly become computationally inefficient or even impossible as the size of the instrument universe and/or the number of Monte Carlo samples increase. We consider solving the CVaR optimization problem directly using a smoothing technique, assuming a continuous loss distribution. We show that, for a large-scale CVaR optimization problem, the smoothing technique is computationally more efficient and is capable of solving larger problems when compared to a standard linear programming software.

2 Minimizing VaR and CVaR for Derivative Portfolios

Derivatives play important roles in both hedging an existing risk and achieving investment goals. Consider the following example. Suppose that the current stock price is $S_0 = \$100$. A writer, of a European at-the-money call on this stock which matures in 10 days, wants to hedge the risk at maturity using the underlying stock and a set of more liquid call options which expire in one, two, three, and six months with strike prices of \$90, \$95, \$100, \$105, and \$110. Specifically, the writer wants to use the underlying stock and the twenty liquid call options to hedge the risk of the short maturity at-the-money call in terms of the expected loss conditional on the loss exceeding, e.g., 95% VaR. To decide how to hedge this risk, the writer needs to solve the following risk minimization problem: given the underlying stock and the 20 liquid options, find the optimal portfolio positions in the 21 instruments so that the CVaR of the hedging portfolio, consisting of the existing short maturity call and the hedging instruments, is minimized.

More generally, we assume that the available instruments $\{V_1, \cdots, V_n\}$ are derived from underlying assets $\{S_1, S_2, \cdots, S_d\}$ which may be correlated; let the random vector $S \in \Re^d$ denote the underlying values. In this paper, we focus on the derivative hedging problem and assume that the available instruments are derivative contracts. Each derivative contract typically depends on a small subset of the underlying assets, e.g., a stock option value may depend only on one risky asset price. There are various types of derivative contracts on each underlying asset, e.g., vanilla calls and puts, exotic contracts such as binary options and barrier options with many new derivative contracts continuously emerging. For each type of option, there can be different contract specifications, e.g., strike prices and maturities, which give rise to many different possible instruments. In general, for a derivative portfolio optimization problem, the total number of instruments n is far greater than the total number of underlyings d.

At any time t, the value of a derivative contract V_i typically depends nonlinearly on the underlying; the exact value depends on the assumed model for the underlying assets and its associated parameters. For simplicity, let us assume that the ith derivative value at time t is a function of the underlying price S denoted as $V_i(S, t)$, $1 \leq i \leq n$, and the values of the instruments in the selection universe are denoted by the column vector $V(S,t) \stackrel{\text{def}}{=} [V_1(S,t), \cdots, V_n(S,t)]$. We note that not all derivative values can be written simply as $V(S_t,t)$ where S_t represents time t underlying asset values. Asian options for example have a strong dependency on the history of the stock price. Let $x \in \Re^n$ represent a portfolio, where x_i denotes the position for the ith instrument V_i and $\Pi^0(S,t)$ denotes the loss at time t of the existing portfolio P^{init}, i.e.

$$\Pi^0(S,t) = P_0^{\text{init}} - P_t^{\text{init}}$$

where P_0^{init} and P_t^{init} denote the initial and time t values of the existing portfolio. For the hedging example of the short maturity at-the-money call for a writer, $P_{\bar{t}}^{\text{init}} = -\max(S-K,0)$ and the loss at maturity is $\Pi^0(S,\bar{t}) = P_0^{\text{init}} + \max(S-K,0)$ where $K = S_0$ is the strike price of the call.

To measure the risk associated with a hedging portfolio, for a given hedging horizon $\bar{t}$, let us consider the loss function:

$$f_{\text{hedge}}(x,S) = \Pi^0(S,\bar{t}) - x^T\left(V(S,\bar{t}) - V(S_0,0)\right). \tag{1}$$

Let $f(x,S)$ denote any specified loss function. Without loss of generality, assume that the random variable $S \in \Re^d$ of underlying asset values at a hedging horizon $\bar{t}$ has a probability density $p(S)$. For a given portfolio x, the probability of the loss not exceeding a threshold α is given by the cumulative distribution function

$$\Psi(x,\alpha) \stackrel{\text{def}}{=} \int_{f(x,S)\le\alpha} p(S)dS. \tag{2}$$

Under the assumption that the probability distribution for the loss has no jumps, $\Psi(x,\alpha)$ is everywhere continuous with respect to α.

Let $0 < \beta < 1$ denote a specified confidence level, e.g., $\beta = 95\%$. The VaR of a portfolio x, for a confidence level β, is given by

$$\alpha_\beta(x) \stackrel{\text{def}}{=} \inf\{\alpha \in \Re : \ \Psi(x,\alpha) \ge \beta\} \tag{3}$$

and CVaR, the conditional expectation of the loss, given the loss is $\alpha_\beta(x)$ or greater, is given by

$$\phi_\beta(x) \stackrel{\text{def}}{=} (1-\beta)^{-1} \int_{f(x,S)\ge\alpha_\beta(x)} f(x,S)p(S)dS. \tag{4}$$

When the cumulative distribution function $\Psi(x,\alpha)$ is everywhere continuous, there exists α (possibly not unique) such that $\Psi(x,\alpha) = \beta$.

The portfolio CVaR optimization problem can be formulated as

$$\min_{x\in\Re^n} \ \phi_\beta(x). \tag{5}$$

More generally, there may be constraints, typically in the form of linear constraints, on the optimal portfolio. Let $X \subset \Re^n$ denote the feasible portfolios, e.g., $X = \{x : \ l \le x \le u\}$ if the only constraints are the bounds on the instrument positions. Thus a CVaR optimization problem can be more generally formulated as

$$\min_{x\in X} \ \phi_\beta(x). \tag{6}$$

2.1 How Well Is the Minimum Risk Derivative Portfolio Defined?

For derivative portfolio optimization problems, we typically have many derivative contracts depending on the same underlying assets. However, the value of each derivative contract typically depends nonlinearly on the underlying values. How well is the risk optimization problem (5) defined?

In order to measure the risk quantitatively, we need to assume that a stochastic model for the change of the underlying assets of all the instruments in a portfolio is given. In addition, we assume that there exist methods for computing the derivative values, such as Black-Scholes formulae, delta-gamma approximations, or Monte Carlo simulations. For simplicity, our illustrative computational results use the Black-Scholes type analytic formulae to compute derivative values.

We now analyze the properties of an optimal CVaR/VaR derivative portfolio. To do this, let us consider delta-gamma approximations for derivative values. For a short time horizon $t > 0$, a delta-gamma approximation can be an acceptably accurate approximation to the derivative value and is often used in risk assessment. In general, the delta-gamma approximation describes the most significant component in the change of the derivative values and can thus provide insight into the nature of the solution. Therefore, let us assume, for instrument i,

$$V_i(S,t) - V_i^0 = \left(\frac{\partial V_i^0}{\partial t}\right)\delta t + \left(\frac{\partial V_i^0}{\partial S}\right)^T(\delta S) + \frac{1}{2}(\delta S)^T\Gamma_i(\delta S). \tag{7}$$

Here the vector $(\delta S) \in \Re^d$ denotes the change in the underlying assets, $\frac{\partial V_i^0}{\partial t}$ denotes the initial theta sensitivity of the ith instrument value to time, $\frac{\partial V_i^0}{\partial S} \in \Re^d$ denotes the initial delta sensitivity of the ith instrument with respect to the underlyings, and $\Gamma_i \in \Re^{d\times d}$ is the Hessian matrix denoting the initial gamma sensitivity of the ith instrument with respect to the underlyings, and δt is change in time.

Let $\frac{\partial V^0}{\partial t}$ and $\frac{\partial V^0}{\partial S}$ denote the initial sensitivities to change in time and underlying risk factors (first order) respectively for all instruments in the decision universe:

$$\begin{aligned}\frac{\partial V^0}{\partial t} &\stackrel{\text{def}}{=} \left[\frac{\partial V_1^0}{\partial t}, \cdots, \frac{\partial V_n^0}{\partial t}\right] \in \Re^n\\ \frac{\partial V^0}{\partial S} &\stackrel{\text{def}}{=} \left[\frac{\partial V_1^0}{\partial S}, \cdots, \frac{\partial V_n^0}{\partial S}\right]^T \in \Re^{n\times d}\end{aligned}$$

Let $(\delta S)^2 \in \Re^d$ be the vector with each entry of (δS) squared. For simplicity of analysis here, let us assume that each instrument depends on a single risky asset. In this case the only non-zero entries in the gamma sensitivity matrix Γ_i are the diagonal entries. Let us collect these nonzero entries in the following matrix below:

$$\Gamma \stackrel{\text{def}}{=} \left[\Gamma_1^{diag}, \cdots, \Gamma_n^{diag}\right]^T \in \Re^{n\times d},$$

where Γ_i^{diag} is the diagonal of Γ_i written as a column vector.

Now let the matrix Λ denote all these sensitivities:

$$\Lambda \stackrel{\text{def}}{=} \left[\left(\frac{\partial V^0}{\partial t}\right), \left(\frac{\partial V^0}{\partial S}\right), \frac{1}{2}\Gamma\right] \in \Re^{n\times(2d+1)}. \tag{8}$$

Then the loss in portfolio value, for a portfolio hedging problem and hedging horizon $\bar{t}$, is

$$f_{\text{hedge}}(x,S) = \Pi^0(S,\bar{t}) - x^T\Lambda \begin{bmatrix} \delta t \\ \delta S \\ (\delta S)^2 \end{bmatrix} \tag{9}$$

where $\delta t = \bar{t}$. If $n > 2d+1$, there exists $z \in \Re^n$, $z \neq 0$, such that $\Lambda^T z = 0$. Therefore, for any θ,

$$f_{\text{hedge}}(x,S) \equiv f_{\text{hedge}}(x+\theta z, S), \quad \text{for any } S$$

Hence the portfolios x and $x+\theta z$ have the same VaR/CVaR under the delta-gamma approximation.

The above analysis leads to following conclusion: The CVaR derivative portfolio hedging problem $\min_{x\in\Re^n} \ \phi_\beta(x)$ based on CVaR with the loss function

$$f_{\text{hedge}}(x,S) = \Pi^0(S,\bar{t}) - x^T\left(V(S,\bar{t}) - V(S_0,0)\right)$$

is ill-posed in the sense that there are an infinite number of solutions, when $n > 2d+1$ and the hedging derivative values are given by delta-gamma approximations. Similarly, the VaR derivative portfolio hedging optimization problem $\min_{x\in\Re^n} \ \alpha_\beta(x)$ is ill-posed.

When the derivative values are computed through more accurate methods, such as analytic formulae or Monte Carlo simulation, the CVaR/VaR hedging optimization problem typically remains ill-posed in the sense that an arbitrarily small perturbation of the data can cause an arbitrarily large perturbation in the solution. The concepts of well-posed and ill-posed problems were first raised by Hadamard at the beginning of the 20th century [11]. Since then ill-posed problems have emerged from many areas of science and engineering, typically in the form of inverse problems.

In [3], a similar analysis is made for a portfolio selection problem. For a portfolio x , we consider the loss function

$$f_{\text{select}}(x,S) = -x^T\left(V(S,\bar{t}) - V(S_0,0)\right), \tag{10}$$

for a portfolio selection problem, where $\bar{t} > 0$ is a given time horizon. For this problem, one wants to choose a portfolio which minimizes the risk, e.g., CVaR, under the requirement that the portfolio has a specified return. Let $(\delta V) \in \Re^n$ denote the change in instrument values at the given time horizon $\bar{t}$, i.e., $(\delta V) = V(S,\bar{t}) - V^0$ where $V^0, V(S,\bar{t}) \in \Re^n$ denote the initial value and value at time $\bar{t}$ respectively. Let us assume that we have a particular budget, for example, an initial \$1 budget for simplification; the budget constraint can be expressed as

$$\left(V^0\right)^T x = 1.$$

(The VaR and CVaR of a portfolio with a budget ρ are simply $\rho \cdot \alpha_\beta(x)$ and $\rho \cdot \phi_\beta(x)$ respectively). Alternatively, x can be interpreted as the ratio of the instrument holdings to the total initial budget for investment, i.e., x_i is the number of units of the ith instrument holding per dollar investment.

The return constraint can be written as

$$\overline{\delta V}^T x = \bar{r}$$

where $\bar{r} \geq 0$ specifies the mean return of the portfolio and $\overline{\delta V} \in \Re^n$ is the expected gain for the instruments in the time horizon $\bar{t}$, i.e., $\overline{\delta V} = \mathbf{E}[(\delta V)]$.

Let X denote the feasible region specified by these two constraints:

$$X = \{x : \left(V^0\right)^T x = 1 \text{ and } \overline{\delta V}^T x = \bar{r}\}.$$

The portfolio selection problem based on the CVaR measure can be formulated as

$$\begin{array}{ll} & \min_{x \in \Re^n} \ \phi_\beta(x) \\ \text{subject to} & \left(V^0\right)^T x = 1 \\ & \overline{\delta V}^T x = \bar{r}. \end{array} \tag{11}$$

From similar analysis, we can deduce that if $n > 2(d+1)+1$, then the optimal CVaR and VaR portfolio selection problems defined by $\min_{x \in X} \phi_\beta(x)$ and $\min_{x \in X} \alpha_\beta(x)$, for any $0 < \beta < 1$, lie in a linear subspace of dimension $n - 2(d+1) - 1$, if the derivative values are given by their delta-gamma approximations. Thus, the CVaR/VaR derivative portfolio selection optimization problem is typically ill-posed.

2.2 Difficulties Due to Ill-posedness

What are the consequences of the ill-posedness of the derivative portfolio optimization problem? Can these difficulties be easily overcome by imposing simple constraints, e.g., bound constraints? Here we investigate these questions by considering the hedging problem of the short maturity at-the-money call option introduced at the beginning of §2.

First, let us discuss how a CVaR optimization problem (6) can be solved. Rockafellar and Uryasev [17] introduce the auxiliary function:

$$F_\beta(x, \alpha) \stackrel{\text{def}}{=} \alpha + (1-\beta)^{-1} \int_{S \in \Re^d} [f(x, S) - \alpha]^+ p(S) dS \tag{12}$$

where

$$[f(x,S) - \alpha]^+ \stackrel{\text{def}}{=} \begin{cases} f(x,S) - \alpha & \text{if } f(x,S) - \alpha > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown [17] that the function $F_\beta(x, \alpha)$ is convex and continuously differentiable with respect to α if the cumulative distribution function $\Psi(x, \alpha)$ is continuous. Moreover, minimizing CVaR over any $x \in X$, where X a subset of $\Re^n$, is equivalent to minimizing $F_\beta(x, \alpha)$ over all $(x, \alpha) \in X \times \Re$, i.e.,

$$\min_{x \in X} \phi_\beta(x) \equiv \min_{(x,\alpha) \in X \times \Re} F_\beta(x, \alpha). \tag{13}$$

The function $F_\beta(x, \alpha)$ is convex with respect to (x, α) and the CVaR function $\phi_\beta(x)$ is convex with respect to x if the loss function $f(x, S)$ is convex with respect to x. If, in addition, X is a convex set, then the minimization problem

$$\min_{(x,\alpha) \in X \times \Re} F_\beta(x, \alpha) \tag{14}$$

is a convex programming problem. The convexity property is appealing since any local minimizer of a convex programming problem is a global minimizer.

The loss function, $f_{\text{hedge}}(x, S)$, of the portfolio hedging problem over the time horizon $\bar{t}$ can be written as $\Pi^0(S, \bar{t}) - (\delta V)^T x$; $f_{\text{hedge}}(x, S)$ is clearly convex with respect to x. We can formulate (14) explicitly as

$$\min_{(x,\alpha)\in X\times\Re} \left(\alpha + (1-\beta)^{-1} \int_{S\in\Re^d} [\Pi^0(S,\bar{t}) - (\delta V)^T x - \alpha]^+ p(S) dS \right) \tag{15}$$

A continuous CVaR optimization problem (15) can be solved approximately using Monte Carlo simulation. Assume that $\{(\delta V)_i\}_{i=1}^m$ are independent samples of the change of the instrument values (δV) over the given horizon. Let $\{(\Pi^0)_i\}_{i=1}^m$ denote the corresponding samples of the loss of the existing portfolio. Then the following piecewise linear optimization problem is an approximation to the continuous optimization problem (15):

$$\min_{(x,\alpha)\in X\times\Re} \left(\bar{F}_\beta(x,\alpha) \stackrel{\text{def}}{=} \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^m [\left(\Pi^0\right)_i - (\delta V)_i^T x - \alpha]^+ \right) \tag{16}$$

To solve this piecewise linear optimization problem, let us consider the following problem,

$$\begin{array}{ll} & \min\limits_{(x,y,\alpha)} \left(\alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^m y_i \right) \\ \text{subject to} & y_i + (\delta V)_i^T x + \alpha \geq (\Pi^0)_i, \;\; i = 1, 2, \cdots, m \\ & x \in X, \;\; y \geq 0 \end{array} \tag{17}$$

Note that, for any solution of (17), it can easily be shown that $y_i = [\left(\Pi^0\right)_i - (\delta V)_i^T x - \alpha]^+$, $1 \leq i \leq m$, will always be satisfied. Thus solving (17) is equivalent to solving (16).

If the feasible region X is specified by a set of linear constraints, problem (17) is a minimization of a linear function subject to linear constraints. This is a linear programming problem which can be solved by standard methods. In §4, we discuss, in greater detail, the computational issues for CVaR minimization by solving an equivalent linear programming problem. In addition we describe a smoothing method for solving the simulation CVaR minimization directly. Here we use an interior point software, MOSEK [1], to solve (17).

In order to illustrate the difficulties that may arise from the ill-posed CVaR optimization problem, let us consider the hedging problem faced by a writer of a short maturity call, who wants to reduce the risk he will face at the maturity of the option by trading more liquid options presently. We assume that the current stock price $S_0 = \$100$, the current Black-Scholes (implied) volatility $\sigma_0 = 20\%$, and the risk free interest $r = 4\%$. For simplicity, we assume that the stock pays no dividend and its expected return is 10% per annum. The at-the-money call expires in 10 trading days (we assume here that there are 252 trading days in a year). The hedging horizon $\bar{t}$ is the maturity of the call and the loss of the existing portfolio is $\Pi^0(S, \bar{t}) = P_0^{\text{init}} + \max(S_t - K, 0)$ where $K = S_0$ is the strike price. Currently the liquid call options on this stock have maturities of one, two, three and six months and strike prices of \$90, \$95, \$100, \$105, and \$110. The hedging instrument universe consists of the stock and the twenty liquid call options. We solve the hedging problem (17) with a lower limit of $l = -100$ and upper limit of $u = 100$ on holding positions:

$$\begin{array}{ll} & \min_{(x,y,\alpha)} \left(\alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} y_i \right) \\ \text{subject to} & y_i + (\delta V)_i^T x + \alpha \geq \left(\Pi^0\right)_i, \; i = 1, 2, \cdots, m \\ & -100 \leq x \leq 100, \; y \geq 0. \end{array} \tag{18}$$

The independent samples of change $\{(\delta V)_i\}_{i=1}^m$ and the loss of the existing portfolio $\{(\Pi^0)_i\}_{i=1}^m$ are computed from

$$S_{\bar{t}} = S_0 e^{\mu \cdot \bar{t} + \sqrt{\bar{t}} \cdot \sigma_0 \cdot \Omega}$$

where $\sigma_0 = 20\%$ and Ω is a standard normal.

Let us first consider the risk improvement that can be achieved from solving the hedging problem (18) in terms of VaR and CVaR. The VaR and CVaR reported in this paper correspond to the VaR and CVaR from Monte Carlo simulations. We compute these risks as follows: for a simulation with m independent samples, a portfolio x has losses, $(\text{loss})_1 \leq \ldots \leq (\text{loss})_m$, and each $(\text{loss})_i$ has a probability of $\frac{1}{m}$. For a confidence level β, we compute the VaR and CVaR as follows: let $i_\beta \leq m$ be the index such that

$$\frac{i_\beta}{m} \geq \beta > \frac{i_\beta - 1}{m}. \tag{19}$$

Then VaR is given by $\alpha_\beta(x) = (\text{loss})_{i_\beta}$ and CVaR is given by

$$\phi_\beta(x) = \frac{1}{1-\beta} \sum_{i=i_\beta+1}^{m} \frac{(\text{loss})_i}{m}. \tag{20}$$

For a confidence level of $\beta = 95\%$ and a time horizon $\bar{t}$ equal to the maturity of the at-the-money call, the VaR of this short maturity call equals $\alpha_\beta(0) = \$5.5291$ and CVaR is $\phi_\beta(0) = \$7.4396$. Let x^* denote the optimal hedging portfolio from (18). The VaR and CVaR associated with x^* are $\alpha_\beta(x^*) = -\$12.7857$ and $\phi_\beta(x^*) = -\$12.6816$, an improvement of 331% and 271% respectively over the VaR and CVaR of the at-the-money short maturity call.

We exclude the ith instrument from the optimal portfolio if $|x_i^*| \leq 10^{-3}$ (and the reported risk corresponds to that of the portfolio after this exclusion). From Figure 1, we observe that the optimal hedging portfolio consists of all 21 instruments, with 15 instrument holdings at either the lower or upper bounds. The number of units traded is $\|x_0^*\|_1 = 1732$ (i.e., a large transaction cost of \$17.32 assuming 1% transaction cost per unit traded). Note that the initial Black-Scholes price for this short maturity call is \$1.67. This indicates that the optimal hedging portfolio is unattractive since it incurs large transaction and management costs. In addition, it will be illustrated in §3.1 that these extreme positions can lead to the more serious problem of potentially magnifying model error.

One might have hoped that adding constraints would eliminate or alleviate the ill-posedness of the problem. We caution that one needs to be careful to ensure that these

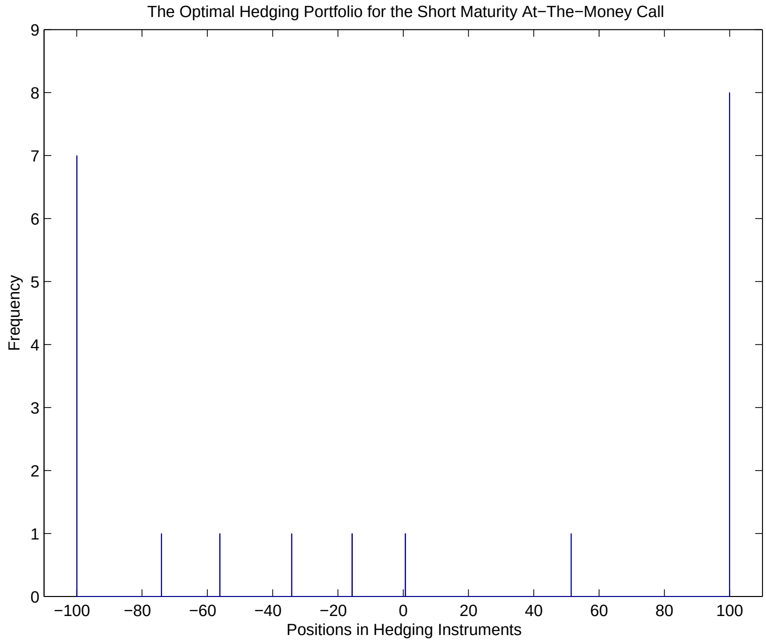

Figure 1: Histogram of Holdings of the Optimal Portfolio x_0^* From (18): $\beta = 0.95$

constraints are meaningful and consistent in the sense that there exist feasible solutions. In addition, the hedging example here illustrates that it is difficult to remove the ill-posedness by simply adding constraints. Indeed the optimization problem with the bound constraints remains ill-conditioned in the sense that there are many different portfolios with similar CVaR; further evidence illustrating this will be provided in §3.1.

We discuss next how to obtain a more desirable optimal hedging portfolio based on CVaR. We note that much of the following discussion is applicable to the VaR optimization problem; however we focus only on the CVaR optimization problem subsequently due to its computational tractability.

3 Regularizing the Derivative CVaR Optimization

The difficulty associated with ill-posed problems is that they are practically underdetermined. Thus it is important to incorporate additional meaningful information about the desired solution in order to stabilize the problem and produce a useful solution. This is often referred to as *regularization* of an ill-posed problem.

For portfolio management, a natural consideration is transaction and management cost; a portfolio, which, in addition to a small CVaR, incurs a small transaction and management cost, is certainly more attractive. One can regard the management cost as proportional to the total number of (non-zero) instrument holdings in a portfolio. Unfortunately, it is difficult to include this explicitly into an optimization formulation since it is computationally challenging to solve the resulting mixed integer program. Our objective is to seek a portfolio which consists of a small number of instruments by minimizing a combination of the CVaR and a suitable cost function.

Let us assume that the cost of holding an instrument is proportional to the magnitude of the instrument positions. Then we seek a portfolio which has a small CVaR risk as well as a proportional cost by solving

$$\min_{x\in X} \left(\phi_\beta(x) + \sum_{i=1}^{n} c_i|x_i|\right) \tag{21}$$

where the CVaR function $\phi_\beta(x)$ is as defined in (4). Here $c \geq 0$ represents the cost as well as the tradeoff between minimizing CVaR and cost.

The cost parameter $c_i \geq 0$ can be interpreted as a measure of relative desirability to exclude the ith instrument from the optimal portfolio. In this sense we can regard c as a combination of the transaction cost and management cost. For any i_0, $1 \leq i_0 \leq n$, assume that all c_i, $,1 \leq i \leq n$ and $i \neq i_0$, are fixed. Then it can be shown that there exists a finite threshold value such that, when c_{i_0} is greater than this threshold, the solution satisfies $x^*_{i_0} = 0$ (in other words the optimal portfolio x^* for (21) excludes the ith instrument). This is because the proportional cost modeling corresponds to an exact penalty function (a technique used for solving a constrained optimization problem). We refer interested readers to [9] for a more detailed discussion on the exact penalty function. Note that if one models the cost as $\sum_{i=1}^{n} c_i x_i^2$ for example, the resulting optimal portfolio typically has few (if any) of its instruments with a small position $|x^*_i|$ (e.g., $|x^*_i| \leq 10^{-3}$). For the quadratic penalty function, the constraint $x^*_{i_0} = 0$ is satisfied only as the penalty parameter c_{i_0} tends to $+\infty$.

To solve (21), we can similarly consider the augmented function $F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|$. It is clear that $F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|$ remains convex and continuously differentiable with respect to α since $\sum_{i=1}^{n} c_i|x_i|$ is convex and has no dependence on α; thus similar analysis of [17] can be applied. Moreover, minimizing the sum of the cost and CVaR of the portfolio x in any subset X of $\Re^n$ is equivalent to minimizing $F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|$ over $(x,\alpha) \in X \times \Re$, i.e.,

$$\min_{x\in X} \left(\phi_\beta(x) + \sum_{i=1}^{n} c_i|x_i|\right) \equiv \min_{(x,\alpha)\in X\times\Re} \left(F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|\right).$$

In addition, the augmented function $F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|$ is convex with respect to (x,α) and $\phi_\beta(x) + \sum_{i=1}^{n} c_i|x_i|$ is convex with respect to x if the loss function $f(x,S)$ is convex with respect to x. Moreover, if X is a convex set, the minimization problem

$$\min_{(x,\alpha)\in X\times\Re} \left(F_\beta(x,\alpha) + \sum_{i=1}^{n} c_i|x_i|\right) \tag{22}$$

is a convex programming problem.

Next we compare the optimal portfolios from the cost model (21) for different cost parameters. Assume that a financial institution is faced with a loss $\Pi^0(S,\bar{t})$ at the horizon $\bar{t}$; the loss function of the portfolio hedging problem is

$$f_{\text{hedge}}(x,S) = \Pi^0(S,\bar{t}) - (\delta V)^T x. \tag{23}$$

For simplicity let us assume that the feasible set X, for the hedging problem, is defined by bound constraints. The simulation CVaR hedging optimization can be equivalently formulated as the following linear programming problem:

$$
\begin{array}{lll}
 & \min\limits_{(x,y,z,\alpha)} & \left(\alpha + \dfrac{1}{m(1-\beta)}\sum\limits_{i=1}^{m} y_i + \sum\limits_{j=1}^{n} c_j z_j\right) \\
\text{subject to} & & y_i + (\delta V)_i^T x + \alpha \geq \left(\Pi^0\right)_i, \quad i = 1, 2, \cdots, m \\
 & & z + x \geq 0, \ \ z - x \geq 0 \\
 & & l \leq x \leq u, \ \ y \geq 0, \ \ z \geq 0
\end{array}
\tag{24}
$$

3.1 Example 1: Hedging a Short Maturity At-The-Money Call

To illustrate the properties of the optimal hedging portfolios with respect to CVaR for different cost parameters, let us consider the same problem of hedging an initial portfolio P^{init} of a short maturity at-the-money call (from a writer's perspective) with more liquid options described in §2.2. We denote the VaR and CVaR of the initial portfolio are denoted by VaR^{init} and $\text{CVaR}^{\text{init}}$.

We consider the cost parameter $c_i = \omega \cdot |\text{CVaR}(0)|$, $1 \leq i \leq n$, where CVaR(0) denotes the optimal CVaR value from (24) but under no cost consideration, i.e., solving (18). Similarly $\text{VaR}(\omega)$ and $\text{CVaR}(\omega)$ denote the VaR and CVaR of the optimal portfolio from (24) with cost parameter ω. We use 20000 Monte Carlo samples for this hedging example.

Table 1 compares the hedging portfolios for various cost parameter values. Again we exclude the ith instrument from the optimal portfolio if $|x_i^*| \leq 10^{-3}$ and the computed risk corresponds to that of the portfolio after this exclusion. We report total number of instruments #Ins with holding position $|x_i^*| > 10^{-3}$. We observe that the optimal hedging portfolio has significantly fewer instruments and smaller total trading positions for larger cost parameters. In addition, significant risk reduction is achieved from all the optimal hedging portfolios.

Let us compare the optimal hedging portfolio under no cost consideration, x_0^*, with the optimal portfolio $x_{0.5\%}^*$ with the cost parameter $\omega = 0.5\%$; here the subscript denotes the value of the parameter ω. The optimal CVaR hedging portfolio $x_{0.5\%}^*$ consists of only 3 hedging instruments, compared to 21 instruments for x_0^*. The top panel in Table 2 displays the VaR and CVaR of the initial portfolio and the optimal portfolios in Table 1. The VaR and CVaR of the portfolio $x_{0.5\%}^*$ are $\alpha_\beta(x_{0.5\%}^*) = \$0.2127$ and $\phi_\beta(x_{0.5\%}^*) = \$0.2168$, an improvement of 96% and 97% respectively over the VaR and CVaR of the existing short maturity at-the-money call. The optimal portfolio $x_{0.5\%}^*$ holds 0.4586 units of the stock, -0.7905 units of the call with strike price \$90 and 1-month expiry, and 1.5832 units of the call with strike \$100 and 1-month expiry (a transaction cost of \$0.03 assuming 1% transaction cost per unit trading). The optimal CVaR hedging portfolio x_0^*, on the other hand, consists of the entire 21 hedging instruments; the VaR and CVaR of this portfolio are $\alpha_\beta(x_0^*) = -\$12.7875$ and $\phi_\beta(x_0^*) = -\$12.6816$, an improvement of 331% and 271% respectively over the VaR and CVaR of the at-the-money short maturity call. The transaction cost for x_0^* is approximately \$17.32.

ω	$\frac{\text{VaR}(\omega)-\text{VaR}(0)}{\text{VaR}(0)}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}(0)}{\text{CVaR}(0)}$	#Ins	$\frac{\text{VaR}(\omega)-\text{VaR}^{\text{init}}}{\text{VaR}^{\text{init}}}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}^{\text{init}}}{\text{CVaR}^{\text{init}}}$	$\Vert x^*\Vert_1$
0	0	0	21	-3.312	-2.705	1732
0.001	0.05331	0.1077	6	-3.189	-2.521	349.3
0.005	1.017	1.017	3	-0.9615	-0.9709	2.832
0.01	1.024	1.024	2	-0.9453	-0.9592	1.700
0.05	1.035	1.036	2	-0.9189	-0.9394	1.254

Table 1: Optimal Hedging Portfolios for a Short Maturity At-The-Money Call: $\beta = 0.95$

Although the hedging portfolio $x^*_{0.5\%}$ significantly improves the risk of the original portfolio and the transaction cost is much more acceptable, the risk associated with $x^*_{0.5\%}$ is much larger than that of x^*_0. Which hedging portfolio should one choose? To help answer this question, we need to keep in mind that any assumed model for the underlying asset is inevitably only an approximation to the actual market price dynamics. For example, let us assume that the Black-Scholes formula gives an accurate pricing formula for the call options. However, the future values of the hedging instruments depend on the future implied volatility at the hedging horizon, $\bar{t} = 10$ trading days, which is unknown at the initial time $t = 0$. Let us assume that the implied volatility at $\bar{t}$ is random with a normal distribution of a mean equal to 20% and a standard deviation of 0.5%, i.e.,

$$\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega, \quad \Omega \in \mathcal{N}(0,1) \tag{25}$$

where $\mathcal{N}(0,1)$ denotes a standard normal; hence, a fairly small error in the future implied volatility estimation is assumed. The underlying risk factors $S \in \Re^d$ now include the stochastic volatility.

The bottom panel in Table 2 provides VaR and CVaR for the optimal portfolios in Table 1 under assumption (25). Both the optimal portfolios x^*_0 (under no cost consideration) and $x^*_{0.1\%}$ (with a small parameter $\omega = 0.1\%$) have significantly *increased* risks compared to that originally faced by the writer! These optimal portfolios have magnified the model error due to their extreme instrument positions. On the other hand, the optimal hedging portfolios $x^*_{0.5\%}$, $x^*_{1\%}$, and $x^*_{5\%}$ (under larger cost parameters) have significantly improved the risk of the writer.

Table 2 indicates that, for a very small cost parameter, the performance of the optimal hedging portfolio is extremely sensitive to the implied volatility error while the performance of the optimal portfolios with larger cost parameters, e.g., $x^*_{0.5\%}$, $x^*_{1\%}$, and $x^*_{5\%}$, are more robust with respect to the model error.

Table 3 displays the expected returns of the hedging portfolios in the hedging horizon $\bar{t}$. We note that the magnitudes of expected returns of all the hedging portfolios are less than .5% ; this is reasonable since these portfolios correspond to minimum risk portfolios under a small cost consideration and large bounds on the holding positions. For this example, the hedging portfolio $x^*_{0.5\%}$ seems to be the most attractive one in terms of the expected return, transaction and management cost, and sensitivity to model error.

Given that the future implied volatility is uncertain in practice, it seems that one should explicitly include the implied volatility uncertainty in the CVaR optimization problem (24)

$\sigma_{\bar{t}} = 20\%$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	-12.7857	-12.1041	0.2127	0.3024	0.4483
CVaR	7.4396	-12.6816	-11.3159	0.2168	0.3039	0.4508

$\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	25.6887	3.7592	0.2346	0.3187	0.4496
CVaR	7.4396	36.1931	9.1392	0.2586	0.3383	0.4597

Table 2: VaR and CVaR Risks at $\bar{t} = 10$ Days for Portfolios in Table 1

	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
$\sigma_{\bar{t}} = 20\%$	11.05	0.16	0.16	0.17	0.00	-0.41
$\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega$	11.05	0.15	0.15	0.17	-0.06	-0.19

Table 3: Expected Return (%) of the Portfolios in Table 1

when determining the optimal hedging strategy. We assume now that the Black-Scholes formula still gives the option price but the future implied volatility is uncertain. Specifically, let us assume that the implied volatility at the hedging time horizon $\bar{t}$ has the distribution (25):

$$\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega$$

where Ω is a standard normal. For simplicity, we have assumed here that the future implied volatility and the underlying prices are independent.

To determine an optimal hedging strategy under the assumption (25) on the implied volatility at the hedging horizon, the future hedging instrument values and the loss of the existing portfolio need to be computed under the stochastic volatility assumption (25). Using Monte Carlo simulation, independent samples of both the underlying $\{(S_{\bar{t}})_i\}_{i=1}^m$ and $\{(\sigma_{\bar{t}})\}_{i=1}^m$ are computed and the corresponding change of hedging instrument values $\{(\delta V)_i\}_{i=1}^m$ and losses of the initial portfolio $\{(\Pi^0)_i\}_{i=1}^m$ are calculated. The CVaR optimization problem (24), under the stochastic volatility assumption (25), uses these computed change $\{(\delta V)_i\}_{i=1}^m$ and loss $\{(\Pi^0)_i\}_{i=1}^m$.

Table 4 compares the hedging portfolios for various cost parameter values. Note that, once again, the total number of instruments in the hedging portfolio decreases as the cost parameter increases. Comparing to Table 1, we see that less risk reduction is achieved when the volatility uncertainty is included in the optimal hedging decision problem.

To illustrate sensitivity of hedging performance to model error, Table 5 provides the VaR and CVaR of the optimal portfolios in Table 4 under different assumptions on the implied volatility at the hedging horizon $\bar{t}$, e.g.,

$$\sigma_{\bar{t}} = 20\% + 0.75\% \cdot \tilde{\Omega}$$

where $\tilde{\Omega}$ is a random variable with a uniform distribution between -1 and 1. We similarly observe that the hedging portfolios under larger cost, e.g, $\omega \geq 0.5\%$, seem to be more robust

ω	$\frac{\text{VaR}(\omega)-\text{VaR}(0)}{\text{VaR}(0)}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}(0)}{\text{CVaR}(0)}$	#Ins	$\frac{\text{VaR}(\omega)-\text{VaR}^{\text{init}}}{\text{VaR}^{\text{init}}}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}^{\text{init}}}{\text{CVaR}^{\text{init}}}$	$\lVert x^* \rVert_1$
0	0	0	21	-3.091	-2.471	1396
0.001	0.02531	0.1338	8	-3.038	-2.274	472
0.005	1.017	1.018	5	-0.9647	-0.9729	3.332
0.01	1.026	1.028	3	-0.9446	-0.9582	1.855
0.05	1.038	1.041	2	-0.9201	-0.9392	1.267

Table 4: Optimal Hedging Portfolios Assuming $\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega$: $\beta = 0.95$

in risk with respect to the model error. Moreover, the hedging portfolios computed under the stochastic volatility assumption are more robust compared to those computed under the constant implied volatility assumption. Specifically, we note (from the second panel in Table 5) the excellent performance of the hedging portfolios x_0^* and $x_{0.1\%}^*$ computed under the stochastic volatility assumption when the implied volatility at $\bar{t}$ turns out to equal the initial implied volatility of 20%. Moreover, from Table 5, for portfolios $x_{0.5\%}^*$, $x_{1\%}^*$, and $x_{5\%}^*$, VaR and CVaR change less significantly with the assumptions on $\sigma_{\bar{t}}$, when compared to the optimal portfolio x_0^* and $x_{0.1\%}^*$.

The computational results for this hedging example illustrate that the optimal hedging portfolios under suitable cost consideration yield more practical hedging strategies in that they significantly improve the existing risk and incur more acceptable transaction and management costs. In addition, their hedging performance is more robust against inevitable model error.

3.2 Example 2: Hedging a Portfolio of Binary Options

We now demonstrate that the proposed optimal CVaR hedging with cost consideration can produce superior hedging portfolios for portfolios of exotic options. Here we illustrate with an example of hedging a portfolio of binary options using liquid vanilla calls. We consider the European binary option which has a discontinuous payoff: a contract pays \$1 if and only if the underlying price is greater than the exercise price at expiry. Because of this discontinuity, a binary option can be difficult to hedge since the delta of the binary option may become large near expiry.

Suppose that a firm currently has a portfolio P^{init} of four short positions of European at-the-money binary call options; each binary option is on a single asset and the four underlying assets are correlated. These options expire in 4, 6, 8, and 10 months on assets 1, 2, 3, and 4 respectively. The firm wants to hedge the risk, for a hedging horizon of one month, using the four underlying assets and 20 vanilla European calls on each asset; the hedging universe now consists of 84 instruments. Again we consider the future loss Π^0 as the initial value of the portfolio P^{init} less its time $\bar{t}$ value .

The covariance matrix of the returns of the four underlying assets is given in Table 6. The initial asset prices S_0, and the expected rates of return of the four assets are given in Tables (7)-(8) . The strike prices of the vanilla calls are $[0.9; 0.95; 1; 1.05; 1.1] \times S_0$ and the

$\sigma_{\bar{t}} = 20\% + 0.5\% \cdot \Omega$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	-11.5633	-11.2707	0.1950	0.3063	0.4415
CVaR	7.4396	-10.9446	-9.4797	0.2015	0.3111	0.4521

$\sigma_{\bar{t}} = 20\%$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	-11.5770	-11.3858	0.1949	0.3047	0.4389
CVaR	7.4396	-11.1926	-9.8598	0.2004	0.3070	0.4429

$\sigma_{\bar{t}} = 20\% + 0.75\% \cdot \tilde{\Omega}$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	-10.8756	-10.7812	0.1955	0.3041	0.4410
CVaR	7.4396	-9.5103	-8.4035	0.2044	0.3099	0.4509

$\sigma_{\bar{t}} = 20\% + 3.5\% \cdot \tilde{\Omega}$						
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$
VaR	5.5291	-4.4417	-6.2919	0.2034	0.3098	0.4555
CVaR	7.4396	1.5188	-0.5423	0.2271	0.3196	0.4707

Table 5: VaR and CVaR Risks at $\bar{t} = 10$ Days for Portfolios in Table 4

options expire in 2, 3, 4, and 6 months.

0.2890	0.0690	0.0080	0.0690
0.0690	0.1160	0.0200	0.0610
0.0080	0.0200	0.0220	0.0130
0.0690	0.0610	0.0130	0.0790

Table 6: Covariance Matrix of Annual Returns

We assume that the risk free interest rate $r = 5\%$, the initial variance

$$\sigma_0^2 = [28.9\%; 11.6\%; 2.2\%; 7.9\%]$$

and the confidence level $\beta = 95\%$. Given that the initial value of the portfolio P^{init} is small and the fact that large positions lead to large transaction cost and sensitivity to model error, we assume, for this example, that the positions in the hedging instruments are restricted by $-1 \leq x \leq 1$.

As before, we assume that the stock prices are described by a geometric Brownian process and the derivatives are priced using a Black-Scholes type analytic formula. Using $m =$

100	50	30	100

Table 7: Initial Asset Prices

0.1091	0.0619	0.0279	0.0649

Table 8: Expected Annual Returns

25000 independent Monte Carlo simulations, we solve the CVaR optimization problem (24) computed using the interior point software MOSEK.

In order to analyze the impact of the cost on the risk measures, we consider the relative differences of VaR and CVaR under different costs with respect to that under no cost and the risk improvement over the existing portfolio. Again we exclude the ith instrument from the hedging portfolio if $|x_i^*| \leq 10^{-3}$.

Let the cost parameter $c_i = \omega \cdot |\mathrm{CVaR}(0)|$ where CVaR(0) denotes the CVaR of the optimal CVaR hedging portfolio under no cost consideration, i.e., solving (24) with $c = 0$. Table 9 displays the properties of the optimal hedging portfolios for different cost parameters. We observe that all these optimal hedging portfolios significantly improve the risk of the initial portfolio; the worst improvement from the hedging portfolio $x_{50\%}^*$ with a parameter $\omega = 0.5$ is approximately 100%. The hedging portfolio contains fewer instruments and incurs a smaller transaction cost as the cost parameter increases. Not surprisingly, the optimal hedging risk increases when the cost parameter increases.

Figure 2 graphs the loss distributions of the existing portfolio and the optimal hedging portfolios x_0^*, $x_{1\%}^*$, and $x_{50\%}^*$ in Table 9. We first observe that the loss distributions of all three optimal hedging portfolios x_0^*, $x_{1\%}^*$, and $x_{50\%}^*$ have flattened right tails as they all minimize a combination of CVaR and cost. In addition, the loss distribution corresponding to a larger cost parameter is right shifted from that corresponding to a smaller cost parameter (a hedging portfolio with a larger cost parameter has a larger risk).

Although significantly improving the risk of the existing portfolio of the binary options, the optimal hedging portfolios in Table 9 have different risks. In order to evaluate the hedging performance more realistically, we now assume that the implied volatility at the hedging horizon $\bar{t}$ is uncertain and is slightly different from the initial implied volatility σ_0. Table 10 displays the expected returns at the hedging horizon $\bar{t}$ and VaR and CVaR of the optimal hedging portfolios under different assumptions on the future implied volatility. First we note that the expected returns of all the optimal portfolios are similar, particularly when $\sigma_{\bar{t}} = \sigma_0$ (top panel in Table 10). If the implied volatility at the hedging horizon $\bar{t}$ equals the initial σ_0, the optimal hedging portfolio x_0^* with no cost consideration has the smallest CVaR. However, if the implied volatility at the hedging horizon has a small error, e.g., $\sigma_{\bar{t}} = \sigma_0 + 2\% \cdot \Omega$, then the optimal hedging portfolio $x_{10\%}^*$ has the least risk. Note that, the optimal hedging portfolios x_0^*, $x_{0.1\%}^*$ $x_{0.5\%}^*$, $x_{1\%}^*$, and $x_{5\%}^*$ actually increase the risk of the

ω	$\frac{\mathrm{VaR}(\omega)-\mathrm{VaR}(0)}{\mathrm{VaR}(0)}$	$\frac{\mathrm{CVaR}(\omega)-\mathrm{CVaR}(0)}{\mathrm{CVaR}(0)}$	#Ins	$\frac{\mathrm{VaR}(\omega)-\mathrm{VaR}^{\mathrm{init}}}{\mathrm{VaR}^{\mathrm{init}}}$	$\frac{\mathrm{CVaR}(\omega)-\mathrm{CVaR}^{\mathrm{init}}}{\mathrm{CVaR}^{\mathrm{init}}}$	$\|\|x^*\|\|_1$
0	0.0000	0.0000	84	-1.8619	-1.6366	73.18
0.001	0.0017	0.0046	64	-1.8604	-1.6336	51.91
0.005	0.0411	0.0483	45	-1.8265	-1.6058	35.98
0.010	0.1072	0.1111	36	-1.7695	-1.5658	27.13
0.050	0.5167	0.5160	22	-1.4165	-1.3081	8.151
0.100	0.9754	0.9767	11	-1.0212	-1.0148	0.284
0.500	0.9955	1.0010	7	-1.0039	-0.9994	0.200

Table 9: Optimal CVaR Hedging Portfolios for $\bar{t} = 1$ Month Assuming $\sigma_{\bar{t}} = \sigma_0$

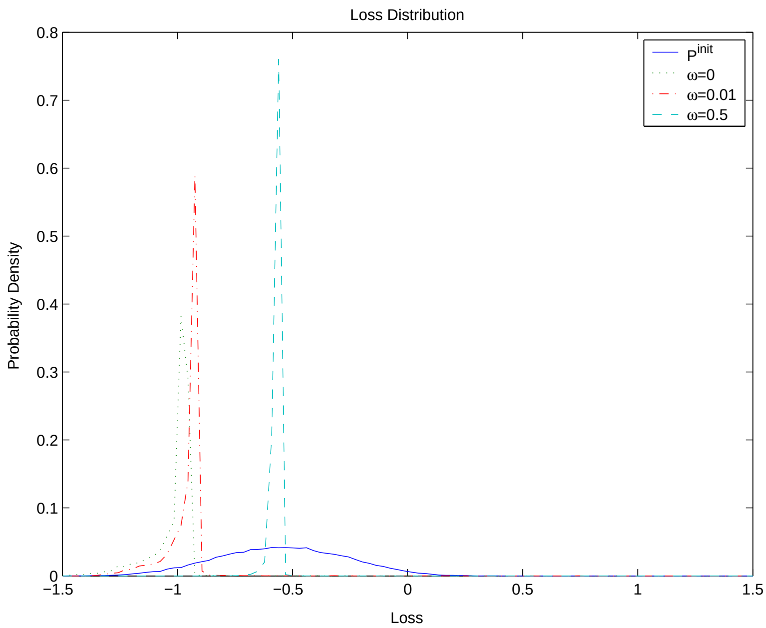

Figure 2: Loss Distributions of the Optimal Hedging Portfolios in Table 9

$\sigma_{\bar{t}} = 20\%$								
	P^{init}	x^*_0	$x^*_{0.1\%}$	$x^*_{0.5\%}$	$x^*_{1\%}$	$x^*_{5\%}$	$x^*_{10\%}$	$x^*_{50\%}$
VaR	0.7515	-0.6477	-0.6466	-0.6211	-0.5783	-0.3130	-0.0101	-0.0029
CVaR	0.9061	-0.5768	-0.5741	-0.5489	-0.5127	-0.2791	-0.0074	0.0006
Return(%)	0.9060	0.4159	0.4158	0.4159	0.4160	0.4167	0.4121	0.4130

$\sigma_{\bar{t}} = \sigma_0 + 2\% \cdot \Omega$								
	P^{init}	x^*_0	$x^*_{0.1\%}$	$x^*_{0.5\%}$	$x^*_{1\%}$	$x^*_{5\%}$	$x^*_{10\%}$	$x^*_{50\%}$
VaR	0.7568	2.0477	1.7765	1.5225	1.4620	0.7382	0.0262	0.0319
CVaR	0.9136	2.7664	2.4298	2.1096	2.0130	1.0482	0.0451	0.0508
Return(%)	0.9832	0.4148	0.4095	0.4064	0.4053	0.3982	0.3807	0.3842

Table 10: VaR/CVaR Risks at $\bar{t} = 1$ Month for Optimal Portfolios in Table 9

ω	$\frac{\text{VaR}(\omega)-\text{VaR}(0)}{\text{VaR}(0)}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}(0)}{\text{CVaR}(0)}$	#Ins	$\frac{\text{VaR}(\omega)-\text{VaR}^{\text{init}}}{\text{VaR}^{\text{init}}}$	$\frac{\text{CVaR}(\omega)-\text{CVaR}^{\text{init}}}{\text{CVaR}^{\text{init}}}$	$\|\|x^*\|\|_1$
0.000	0.0000	0.0000	84	-1.3825	-1.2448	71.515
0.001	0.0016	0.0022	79	-1.3831	-1.2443	65.134
0.005	0.0677	0.0656	59	-1.3566	-1.2288	41.893
0.010	0.1357	0.1354	48	-1.3305	-1.2117	32.036
0.050	0.6791	0.6615	26	-1.1227	-1.0829	4.0961
0.100	0.7671	0.7734	21	-1.0891	-1.0555	2.6552
0.500	1.0338	1.0987	14	-0.9871	-0.9758	0.3643

Table 11: Optimal CVaR Hedging Portfolios for $t = 1$ Month Assuming $\sigma_{\bar{t}} = \sigma_0 + 2\% \cdot \Omega$

existing portfolio P^{init}. Similar to the previous hedging example of the short maturity at-the-money call, hedging performance is very sensitive to model error when the cost parameter is small. This shows the importance of analyzing sensitivity of the hedging performance to model error before actually adopting a hedging strategy.

We similarly investigate the performance of the optimal hedging portfolios when the implied volatility uncertainty is explicitly taken into account in the risk minimization formulation. Table 11 presents the properties of the optimal hedging portfolios computed under the assumption that the implied volatility at the hedging horizon $\sigma_{\bar{t}} = \sigma_0 + 2\% \cdot \Omega$ where Ω is now a 4-dimensional standard normal.

Comparing x^*_0 in Table 11 with x^*_0 in Table 9, we note that the risk reduction is smaller when the future implied volatility is uncertain. In addition, the optimal hedging portfolio x^*_0, assuming the future implied volatility is uncertain, has a smaller total trading positions which leads to a smaller transaction cost.

Table 12 presents, for the optimal portfolios in Table 11, the in-the-sample risk (the implied volatility is the same as that assumed initially) and out-of-the-sample risk when the implied volatility at the hedging horizon has a different distribution. Here $\tilde{\Omega}$ denotes a 4-

$\sigma_{\bar{t}} = \sigma_0 + 2\% \cdot \Omega$								
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$	$x_{10\%}^*$	$x_{50\%}^*$
VaR	0.7568	-0.2895	-0.2899	-0.2699	-0.2502	-0.0929	-0.0674	0.0098
CVaR	0.9136	-0.2237	-0.2232	-0.2090	-0.1934	-0.0757	-0.0507	0.0221
Return(%)	0.9832	0.4038	0.4022	0.4002	0.3999	0.4074	0.4071	0.3805

$\sigma_{\bar{t}} = \sigma_0$								
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$	$x_{10\%}^*$	$x_{50\%}^*$
VaR	0.7515	-0.3239	-0.3259	-0.3022	-0.2776	-0.1027	-0.0772	-0.0150
CVaR	0.9061	-0.2957	-0.2975	-0.2746	-0.2529	-0.0926	-0.0662	-0.0085
Return(%)	0.9060	0.4143	0.4143	0.4142	0.4145	0.4187	0.4189	0.4152

$\sigma_{\bar{t}} = \sigma_0 + 1.5\% \cdot \tilde{\Omega}$								
	P^{init}	x_0^*	$x_{0.1\%}^*$	$x_{0.5\%}^*$	$x_{1\%}^*$	$x_{5\%}^*$	$x_{10\%}^*$	$x_{50\%}^*$
VaR	0.7497	-0.3163	-0.3177	-0.2928	-0.2679	-0.0847	-0.0575	0.0009
CVaR	0.9070	-0.2812	-0.2823	-0.2588	-0.2369	-0.0702	-0.0423	0.0078
Return(%)	0.9175	0.4478	0.4490	0.4261	0.3887	0.1292	0.0961	0.0230

Table 12: VaR/CVaR Risks at $\bar{t} = 1$ Month for Portfolios in Table 11

dimensional random variable with a uniform distribution in the interval $[-1, 1]$. We observe again that the optimal hedging portfolios obtained assuming the implied volatility at the hedging horizon $\bar{t}$ is stochastic are more robust against model error.

4 Minimizing CVaR Efficiently

The CVaR optimization problem (16) is a piecewise linear minimization problem subject to linear constraints. As discussed previously, this minimization problem arising from Monte Carlo simulation can be equivalently formulated as a *linear program (LP)*: minimizing a linear objective function subject to linear equality and inequality constraints. Specifically, for the portfolio hedging problem under bound constraints, we need to solve the linear programming problem (18)

$$\begin{array}{ll} \displaystyle\min_{(x,y,z,\alpha)} & \left(\alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^{m} y_i + \sum_{j=1}^{n} c_j z_j \right) \\ \text{subject to} & y_i \geq \left(\Pi^0\right)_i - (\delta V)_i^T x - \alpha, \quad \text{for } i = 1, \ldots, m \\ & z - x \geq 0, \ z + x \geq 0 \\ & l \leq x \leq u, \ y \geq 0, \ z \geq 0 \end{array}$$

Linear programming is the simplest constrained optimization problem and it is a problem for which the optimization methods have been most successful. Since its formulation in the

1930s and the development of the simplex algorithm by Dantzig [8] in the 1940s, the linear programming problem has been successfully applied to many practical application areas including economics, finance, and engineering. In 1984, Karmarkar [12] proposed a projective affine scaling method for linear programming problems, which has an appealing theoretical property of polynomial complexity. This has renewed greater interest in the theoretical analysis and computational investigation of a new type of method, the interior point method, which has proven to be successful in solving large-scale linear programming problems. Indeed, efficient and reliable software exist for solving large-scale linear programming problems; it is thus not surprising that the linear programming method has been the proposed method to solve a CVaR minimization problem [17, 18].

Is a linear programming method an efficient way to solve a CVaR optimization problem? For a derivative portfolio problem, a CVaR minimization problem can have a large number of instruments and a large number of simulations are typically required to obtain a sufficiently accurate solution. Is a linear programming approach capable of solving such large-scale problems?

4.1 Efficiency for CVaR Minimization Using an LP Approach

In order to understand whether the linear programming method is computationally efficient for solving a CVaR minimization problem, we need to briefly examine the different linear programming methodologies.

Consider a linear programming problem written in its standard form

$$
\begin{array}{rl}
 & \min\limits_{z \in \Re^N} \; c^T z \\
\text{subject to} & Az = b \\
 & z \geq 0
\end{array}
$$

where A is an $M \times N$ full row-rank matrix. Note that any linear program can be converted into this standard form by introducing additional slack variables and artificial variables.

A simplex method computes a solution in a finite number of iterations by following a path from vertex to vertex along the edges of the polyhedron representing the feasible region (defined by linear equality and inequality constraints). For a linear program with N variables, each iteration of a simplex method performs $O(N^2)$ computations; typically the method requires a large number of iterations (roughly between $2M$ to $3M$). An interior point method, on the other hand, produces an infinite sequence of approximations which converge to a solution in the limit. Interior point methods are shown to have polynomial complexity. They require $O(N^3)$ computation per iteration and the number of iterations can be bounded by $O(\sqrt{N}L)$ where L is the input length for integer data. For the CVaR portfolio optimization problem, a potential advantage of the simplex method is its ability to use a warm start when a good starting point is available. Generating a starting point, on the other hand, is an important part of an interior point method. It is not clear how a standard interior point method would utilize a warm start, if at all possible. We consider in this paper CPLEX, a simplex method software and MOSEK, an interior point method software.

The linear programming problem (24), corresponding to an n-instrument, m-scenario, CVaR simulation problem, has $(m + 2n + 1)$ variables and more than $m + n$ linear con-

	MOSEK (cpu sec)			CPLEX (cpu sec)		
	# of instruments being considered					
# scenarios	8	48	200	8	48	200
10000	11.07	61.96	1843.90	53.68	427.97	2120.84
25000	30.02	162.13	14744.64	351.44	2345.43	9907.99
50000	43.62	642.24	-	1673.82	9296.98	-

Table 13: CPU time for standard LP methods: $\beta = 0.99$

straints. With the number of scenarios typically more than 10000, this is a large scale linear programming problem. However, it is a large-scale *sparse* problem since the constraint matrix is a sparse matrix (i.e., it has many zero entries) with an m-by-n dense block (corresponding to the scenarios $\left\{(\delta V)_i^T\right\}_{i=1}^m$). We refer interested readers to [6] for further discussions on efficiently solving large sparse optimization problems. For a CVaR minimization problem with a small number of instruments (and scenarios), a linear programming method approach may be effective. Nonetheless, the computational cost and memory requirement for solving a CVaR problem by a linear programming approach can quickly become prohibitive as the number of scenarios and/or instruments becomes large.

Table 13 illustrates how the cpu time grows with the number of scenarios and the number of instruments for portfolio CVaR optimization problems. With 200 instruments and more than $25,000$ scenarios, a significant amount of the elapsed time is spent in swapping relevant data in and out of the cache memory. With 200 instruments and $50,000$ scenarios, the elapsed time is significantly longer than that of the 48 instrument example, with the memory swapping dominating the elapsed time; the entry is marked by " - " in the table. The comparison is made between CPLEX version 1.3, which implements a simplex method and the MOSEK Optimization Toolbox for MATLAB version 6 (for Solaris Sparc) which implements an interior point method. The problems are implemented in MATLAB version 6.1 and run on a Sun Sparc Ultra-2 machine.

Table 13 clearly illustrates that, using the standard linear programming software, the computational cost as well as the memory requirements quickly become prohibitive as the number of scenarios and the number of instruments increase.

4.2 A Smoothing Technique for CVaR Minimization

We now investigate the computational efficiency issues for the CVaR portfolio optimization problem assuming that the loss is computed using methods such as analytic formulae and Monte Carlo simulation. As an alternative to linear programming approach, we investigate an computationally efficient method which exploits properties of the CVaR optimization problem. The proposed method is applicable under the assumption that the loss distribution is continuous.

We now describe the key observation which leads to our proposed method. Suppose that we are interested in solving a CVaR optimization problem with a continuous distribution through simulation. Intuitively, the piecewise linear objective function

$$\bar{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [\left(\Pi^0\right)_i - (\delta V)_i^T x - \alpha]^+$$

in the simulation problem increasingly approaches the continuously differentiable function $F_\beta(x,\alpha)$. When a large number of scenarios are used, the objective function resembles a smooth function. In order to develop an efficient computational method for the CVaR optimization problem, we explicitly exploit this observation.

Let us approximate the piecewise linear function $\max(0,z)$ by a piecewise quadratic approximation $\rho_\epsilon(z)$: given a resolution parameter $\epsilon > 0$,

$$\rho_\epsilon(z) \stackrel{\text{def}}{=} \begin{cases} z & \text{if } z \geq \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & \text{if } -\epsilon \leq z \leq \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Instead of the piecewise linear approximation $\bar{F}_\beta(x,\alpha)$, we consider the continuously differentiable piecewise quadratic approximation $\tilde{F}_\beta(x,\alpha)$ approximation to $F_\beta(x,\alpha)$:

$$\tilde{F}_\beta(x,\alpha) \stackrel{\text{def}}{=} \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} \rho_\epsilon\left(\left(\Pi^0\right)_i - (\delta V)_i^T x - \alpha\right) \tag{27}$$

To graphically illustrate, let us consider the function $g(\alpha) = \mathbf{E}([S-\alpha]^+)$ assuming that S is a standard normal. Figure 3 graphically illustrates the accuracy and smoothness of the approximations

$$\frac{1}{m} \sum_{i=1}^{m} [S_i - \alpha]^+$$

and $\frac{1}{m}\sum_{i=1}^m \rho_\epsilon(S_i - \alpha)$ as compared to $g(\alpha)$; the top subplot is for $m = 3$ and the bottom subplot is for $m = 10,000$. It can be observed that, as the number of scenarios m increases, the function $\frac{1}{m}\sum_{i=1}^m [S_i - \alpha]^+$ appears smoother and the difference between $\frac{1}{m}\sum_{i=1}^m [S_i - \alpha]^+$ and $\frac{1}{m}\sum_{i=1}^m \rho_\epsilon(S_i - \alpha)$ becomes smaller.

Using the smooth approximation, we solve the following continuously differentiable piecewise quadratic convex programming problem

$$\begin{array}{ll} \min\limits_{(x,\alpha)} & \left(\tilde{F}_\beta(x,\alpha) + \sum_{j=1}^{n} c_j |x_j|\right) \\ \text{subject to} & l \leq x \leq u \end{array} \tag{28}$$

where the approximation $\tilde{F}_\beta(x,\alpha)$ (to the piecewise linear function $\bar{F}_\beta(x,\alpha)$) is a continuously differentiable function. Note that, for (24), each scenario introduces an additional variable (and constraint) in its equivalent linear program formulation. Here the minimization problem has $(n+1)$ independent variables and its equivalent nonlinear program formulation only has $O(n)$ independent variables and constraints.

An optimization method for a convex nonlinear programming problem (28) typically generates an infinite sequence of approximations converging to a solution. At each iteration,

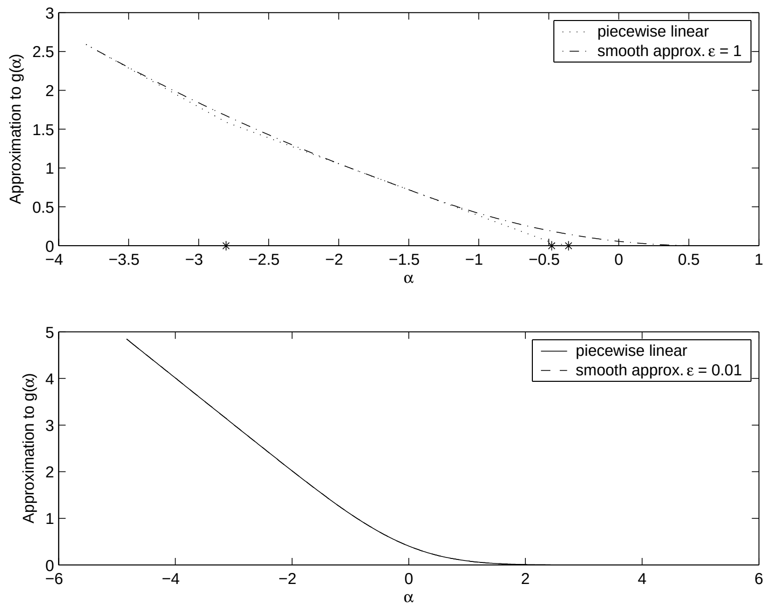

Figure 3: Approximations to $g(\alpha) = \mathbf{E}([S - \alpha]^+)$

	MOSEK (cpu sec)			Smoothing (cpu sec)		
	# of instruments being considered					
# scenarios	8	48	200	8	48	200
10000	6.47	42.04	4244.30	2.45	16.78	419.52
25000	33.50	98.91	10784.10	5.37	35.48	838.15
50000	36.01	318.72	-	9.90	62.08	2080.16

Table 14: CPU times for MOSEK vs. Smoothing: $\beta = 0.99$

however, this approach typically requires a function and a gradient evaluation and $O(n^3)$ linear algebraic operations. The function/gradient evaluation costs $O(mn)$. If the exact second order derivative of the objective function is computed, then the Hessian calculation in the worst case is $O(nk^2)$ where k is the total number of scenarios satisfying $|-(\delta V)_i^T x - \alpha| \le \epsilon$. Given that CVaR optimization minimizes the tail loss with a typical confidence level of $\beta \ge 0.9$, k is usually very small relative to m.

Table 14 makes a comparison between the cpu times of the smoothing technique and the interior point method software MOSEK. The portfolios are the same as those used in Table 13. The smoothing algorithm is based on an algorithm discussed in [7] and is implemented in MATLAB v6.1. The comparison is made on a Sun Sparc Ultra-5_10 machine. We observe that the smoothing technique is much more efficient than the interior point method with up to a 1186% efficiency speedup. In addition, the 200 instruments and 50000 scenario example can now be solved in less than 35 cpu minutes with the smoothing technique due to less memory requirement and better computational efficiency.

In addition to being more computationally efficient, the smoothing technique also yields fairly accurate solutions. We refer interested readers to [3] for more detailed discussions on the comparison of the accuracy and the roles of the resolution parameter ϵ on the accuracy and efficiency of the smoothing technique.

5 Concluding Remarks

In this paper we analyze the ill-posedness of the derivative portfolio risk minimization problem with CVaR and VaR as the choice of risk measures. We illustrate that these minimization problems are typically ill-posed for derivative portfolios. In particular, we have shown that there typically exist an infinite number of portfolios with the same VaR and CVaR when the derivative values are computed through delta-gamma approximations. When the derivative values are computed using more accurate methods such as Black-Scholes formulae and Monte Carlo techniques, the derivative optimal CVaR or VaR problem typically remains ill-posed.

We illustrate that one may not be able to remove the ill-conditioned nature of the CVaR/VaR optimization problem by simply adding constraints. When simple bound constraints are imposed on the instrument holdings, the optimal CVaR derivative hedging portfolio typically has a large number of non-zero instrument holdings (mostly at their bounds). This type of optimal portfolio may not be desirable and can be problematic since it may entail large management and transaction costs. More importantly, the optimal derivative hedging portfolio tends to magnify the modeling error due to extreme holding positions.

We propose to include a proportional cost in the CVaR optimization problem to regularize this ill-posed problem. We illustrate that minimizing CVaR hedging risk together with this cost model can produce more desirable derivative hedging portfolios. Specifically, in addition to reducing the existing risk, the optimal derivative hedging portfolio under suitable cost consideration incurs a smaller transaction cost. It also consists of a significantly smaller number of instruments. In addition, the optimal hedging portfolio under a larger cost is more robust with respect to model error. We demonstrate the importance of analyzing sensitivity of the hedging performance to model error and significance of explicitly including volatility unceratainty in CVaR risk minimization.

We describe a computationally efficient method for solving a simulation based CVaR optimization problem by exploiting the fact that the objective function in the simulation CVaR optimization problem approaches a continuously differentiable function as the number of scenarios increases to infinity. The efficiency and accuracy of this method for solving a CVaR optimization problem are illustrated computationally in greater detail in [3]. In addition, more evidence is provided in [3] for the superior performance of optimal derivative portfolios under cost consideration in the context of portfolio selection problems.

Finally, we remark that the proposed CVaR optimization with cost consideration is applicable to a wide range of derivative portfolio optimization problems including American options and exotic options.

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